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Stochastic Processes and their Applications 118 (2008) 1552–1585
stochastic processes and their applications
www.elsevier.com/locate/spa

Renormalization and convergence in law for the derivative of intersection local time in $\mathbf{R}^2$

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Received 15 August 2006; received in revised form 6 April 2007; accepted 9 October 2007
Available online 13 October 2007

Abstract

In this paper we will examine the derivative of intersection local time of Brownian motion and symmetric stable processes in $\mathbf{R}^2$. These processes do not exist when defined in the canonical way. The purpose of this paper is to exhibit the correct rate for renormalization of these processes.

Keywords: Brownian motion; Local time; Intersection local time; Sample path properties

1. Introduction

Let B_t be a Brownian motion in $\mathbf{R}^d$, and let

$$\alpha_\varepsilon(T) = \int_0^T \int_0^t f_\varepsilon(B_s - B_u)\mathrm{d}s\mathrm{d}t, \tag{1.1}$$

where f_ε denotes the Gaussian density function on $\mathbf{R}^d$ with variance ε. If $\alpha_\varepsilon(T)$ converges to a process as $\varepsilon \longrightarrow 0$ we denote this process α_T and call it the *intersection local time* (henceforth abbreviated as ILT).

In one dimension the ILT does exist, as can be seen easily by using the occupation times formula (see, e.g. [7]). In dimension 2 it does not exist as defined above, as $\alpha_\varepsilon(T)$ blows up as $\varepsilon \longrightarrow 0$ due to the set $\{s = u\}$. In [8], however, Varadhan showed that $\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]$ does converge in law to a process, which is referred to as the *renormalized intersection local time*.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2007.10.003

This process was originally considered due to its relevance to quantum field theory (see [8]), but it has found several other uses, for example in Edwards' work on polymers (see [2]) and Le Gall's work on Wiener sausages (see [4,5]). In [9], Yor proved that in $d = 3$

$$\left\{ \frac{1}{\sqrt{\log(1/\varepsilon)}} \left(\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]\right), T \geq 0 \right\} \tag{1.2}$$

converges in law as $\varepsilon \longrightarrow 0$ to the process $\{\frac{1}{\sqrt{2\pi}} B_T, T \geq 0\}$, where B_T is a one-dimensional Brownian motion. This theorem inspired a similar result from Rosen in [6] involving symmetric stable processes. Rosen considered the process

$$\alpha_\varepsilon(T) = \int_0^T \int_0^t f_\varepsilon(X_t - X_s) \mathrm{d}s \mathrm{d}t, \tag{1.3}$$

where now X is a symmetric stable process of index β, and f_t denotes the density of X_t. Rosen proved that if $4/3 < \beta \leq 2$, then $\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]$ converges pathwise as $\varepsilon \longrightarrow 0$ to a finite random variable. If $\beta = 4/3$, then

$$\left\{ \frac{1}{\sqrt{\log(1/\varepsilon)}} \left(\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]\right), T \geq 0 \right\} \tag{1.4}$$

converges in law as $\varepsilon \longrightarrow 0$ to $\{k(\beta) B_T, T \geq 0\}$ where $k(\beta)$ is a constant which depends on β. Similarly, if $1 < \beta < 4/3$, then

$$\{\varepsilon^{2/\beta - 3/2} \left(\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]\right), T \geq 0\} \tag{1.5}$$

converges in law as $\varepsilon \longrightarrow 0$ to $\{k(\beta) B_T, T \geq 0\}$ where $k(\beta)$ is a constant depending on β. When $\beta = 2$, X is Brownian motion, and this gives a different proof of Varahdan's renormalization. The method employed by Rosen in proving this also gives an alternate proof of Yor's result in 3 dimensions. In [7] Rosen introduced the notion of the derivative of the intersection local time of Brownian motion in $\mathbf{R}^1$. It is defined as

$$\alpha'(T) = \lim_{\varepsilon \longrightarrow 0} \int_0^T \int_0^t f'_\varepsilon(B_t - B_s) \mathrm{d}s \mathrm{d}t, \tag{1.6}$$

provided the limit exists. Formally, we can write

$$\alpha'(T) = \int_0^T \int_0^t \delta'(B_t - B_s) \mathrm{d}s \mathrm{d}t. \tag{1.7}$$

Rosen was able to show that this integral converges as $\varepsilon \longrightarrow 0$, and proved an occupation time formula, as well as some other facts about α'_t. This paper deals with the derivative of ILT in 2 dimensions. In two dimensions, we will use f'_ε to denote $\frac{\delta}{\delta x} f_\varepsilon$. We let

$$\alpha'_\varepsilon(T) = \int_0^T \int_0^t f'_\varepsilon(B_t - B_s) \mathrm{d}s \mathrm{d}t. \tag{1.8}$$

Our main result is that $\alpha'_\varepsilon(T)$ does not converge as $\varepsilon \longrightarrow 0$. We will prove that the asymptotic behavior as $\varepsilon \longrightarrow 0$ is very similar to that which occurs for the ILT in 3 dimensions as discovered by Yor. In particular, our main theorem is

Theorem 1. $\{(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T), T \geq 0\}$ *converges in law to* $\{\frac{\sqrt{5}}{\pi 8\sqrt[4]{2}} W_T, T \geq 0\}$ *as* $\varepsilon \longrightarrow 0$*, where* W_T *is a one-dimensional Brownian motion.*

Remark. f'_ε is an odd function, so $E[\alpha'_\varepsilon(T)] = 0$, which is why we need not subtract the expectation to obtain convergence, as was required in the theorems of Yor, Varadhan, and Rosen.

We also will prove an analogous theorem about symmetric stable processes. We let X_t be a symmetric stable process of index β with $1 < \beta < 2$, let f_t be the density of X_t, and let $f'_t = \frac{\delta}{\delta x} f_t$. Again we will consider

$$\alpha'_\varepsilon(T) = \int_0^T \int_0^t f'_\varepsilon(X_t - X_s)\mathrm{d}s\mathrm{d}t \tag{1.9}$$

and we will prove the following:

Theorem 2. $\{\varepsilon^{3/\beta-3/2}\alpha'_\varepsilon(T), T \geq 0\}$ *converges in law to* $\{c(\beta)W_T, T \geq 0\}$ *as* $\varepsilon \longrightarrow 0$ *where* W_T *is a one-dimensional Brownian motion and* $c(\beta)$ *is given by*

$$\frac{1}{2\sqrt{2}\pi^2}\left(\int\int \frac{1}{|p|^\beta}\frac{1}{|q|^\beta}\frac{1}{|p+q|^\beta}\mathrm{e}^{-(|p|^\beta+|q|^\beta)}p_1q_1\mathrm{d}p\mathrm{d}q \right.$$
$$\left. + \int\int \frac{1}{|p|^{2\beta}}\frac{1}{|p+q|^\beta}\mathrm{e}^{-(|p|^\beta+|q|^\beta)}p_1q_1\mathrm{d}p\mathrm{d}q\right). \tag{1.10}$$

Included in the proof is the definition of the second integral in the definition of $c(\beta)$. This integral does not converge absolutely for $\beta \geq 3/2$, so we must clarify what it means.

The paper is organized as follows. Section 2 gives the outline of the proof. Sections 3 and 4 prove the bulk of the required technical details. Section 5 wraps up the proof of Theorem 1. Section 6 deals with the symmetric stable case, and gives the proof of Theorem 2.

2. Outline of the proof

The outline of the proof of Theorem 1 follows closely the proofs of Theorems 1, 2, and 3 given in [6], though the details, given in Sections 3 and 4, are quite different. The reader may refer to that paper to see a slightly different presentation of the ideas of this section. We will show first that the moments of $\alpha'_\varepsilon(T)(\log(1/\varepsilon))^{-1}$ converge to the moments of a Brownian motion $\frac{\sqrt{5}}{\pi\sqrt{128\sqrt{2}}}$ times. Recall that

$$f_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{d/2}}\mathrm{e}^{\frac{-|x|^2}{2\varepsilon}}. \tag{2.1}$$

We will express this via the Fourier transform in a form which is easier for us to use:

$$f_\varepsilon(x) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^2}\mathrm{e}^{\mathrm{i}px-\varepsilon p^2}\mathrm{d}^2p. \tag{2.2}$$

This formula employs two abbreviations of the notation. x and p are both vectors in $\mathbf{R}^2$, so px denotes $p \cdot x$, and p^2 denotes $|p|^2$. We will use these abbreviations throughout the paper in the hope of reducing the clutter of many of the formulas. We will also use p^β to denote $|p|^\beta$ in the later part of the paper, (2.2) gives

$$f'_\varepsilon(x) = \frac{\mathrm{i}}{(2\pi)^2}\int_{\mathbf{R}^2} p_1 \mathrm{e}^{\mathrm{i}px-\varepsilon p^2}\mathrm{d}^2 p, \tag{2.3}$$

where $p = (p_1, p_2)$. We let D_T denote the triangle $\{(s_i, t_i)|0 \le s_i \le t_i \le T\}$. We then have

$$E(\alpha'_\varepsilon(T)^n) = \frac{\mathrm{i}^n}{(2\pi)^{2n}}\int_{(\mathbf{R}^2)^n}\int_{D_T^n} \mathrm{e}^{-\varepsilon\sum_j p_j^2}\prod_{j=1}^n p_{j,1} E\left[\prod_{j=1}^n \mathrm{e}^{\mathrm{i}p_j(X_{t_j}-X_{s_j})}\right] \times \prod_{j=1}^n \mathrm{d}s_j \mathrm{d}t_j \mathrm{d}^2 p_j. \tag{2.4}$$

This is obtained by combining n copies of the integral (1.8) which defines $\alpha'_\varepsilon(T)$, using the definition (2.2) of f_ε. Now, if n is odd, then the integrand is an odd function of p, and the expectation is therefore 0. Since all odd moments of Brownian motion are 0, we need only show that the even moments converge to the right values. The $2n$th moment of Brownian motion at time T is $\frac{(2n)!}{2^n n!}T^n$, so we will show that

$$E[\alpha'_\varepsilon(T)^{2n}] = \frac{(2n)!}{2^n n!}\left(\frac{\sqrt{5}}{\pi\sqrt{128\sqrt{2}}}(\log(1/\varepsilon))\right)^{2n} T^n + o(\log(1/\varepsilon))^{2n}. \tag{2.5}$$

$E[\alpha'_\varepsilon(T)^{2n}]$ is given by (2.4) with n replaced by $2n$. In order to deal with the integral on the right-hand side of (2.4), we would like to factor the expectation in the integrand using the independence of the Brownian increments. This factoring, however, will in general depend on the ordering of s_j's and t_j's, and we will therefore split the set D_T^{2n} into many regions, each corresponding to an ordering of s_j's and t_j's. We then proceed separately over each region. In each region, independence allows us to factor $E[\prod_{j=1}^n \mathrm{e}^{\mathrm{i}p_j(X_{t_j}-X_{s_j})}]$ into the product of M expectations, where M is the number of components in the set $\bigcup_j [s_j, t_j]$ in that region of D_T^{2n}. Following [6], we will say that a component consisting of m intervals $[s_j, t_j]$ is of *order m*. Suppose for the time being that we are considering a region where $\bigcup_j [s_j, t_j]$ consists of n components of order 2. For now we will hold fixed the initial points (which are necessarily s values) of the n components, let us call them $r_1 < \cdots < r_n$. In this case we will show (Section 3) that, upon integrating all variables in (2.4) other than $\{r_1, \ldots, r_n\}$, each of the n components contribute

$$\left(\frac{5}{\pi^2 128\sqrt{2}}(\log(1/\varepsilon))\right)^2 + o(\log(1/\varepsilon))^2. \tag{2.6}$$

The contribution of each configuration with n components of order 2 to (2.4) is therefore

$$\begin{aligned}
&\left(\frac{5}{\pi^2 128\sqrt{2}}(\log(1/\varepsilon))^2 + o(\log(1/\varepsilon)^2)\right)^n \int_{0\le r_1\le\cdots\le r_n\le T} \mathrm{d}r_1\cdots\mathrm{d}r_n \\
&= \frac{T^n}{2^n n!}\left(\frac{\sqrt{5}\sqrt{2}}{\pi\sqrt{128\sqrt{2}}}(\log(1/\varepsilon))\right)^{2n} + o(\log(1/\varepsilon))^{2n} \\
&= \frac{T^n}{2^n n!}\left(\frac{\sqrt{5}}{\pi 8\sqrt[4]{2}}(\log(1/\varepsilon))\right)^{2n} + o(\log(1/\varepsilon))^{2n},
\end{aligned} \tag{2.7}$$

where we have used the identity

$$\int_{0\le r_1\le\cdots\le r_n\le T} \mathrm{d}r_1\cdots\mathrm{d}r_n = T^n/n. \tag{2.8}$$

There are $(2n)!$ different ways to choose the ordered set $\{r_1 < \cdots < r_n\}$ from the set $\{s_1, \ldots, s_{2n}\}$, and each contributes (2.7), so the total contribution from regions of this type is

$$\frac{(2n)!}{2^n n!}\left(\frac{\sqrt{5}}{\pi 8\sqrt[4]{2}}(\log(1/\varepsilon))\right)^{2n} T^n + o(\log(1/\varepsilon))^{2n}, \tag{2.9}$$

which is exactly what we were aiming for. We must therefore show that the contribution from all regions which have other than n components of order 2 is $o(\log(1/\varepsilon))^{2n}$. We will do this in Section 4 by showing that each component of order m with $m \ge 3$ contributes $o(\log(1/\varepsilon))^m$. Note that any component of order 1, or indeed of any odd order, will in fact contribute 0, since the integrand is an odd function. It will follow from all of this that the moments of $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T)$ converge to the right values. We will prove that the processes $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T)$ are tight in Section 5, so that there is a limiting process which has all of the same moments as $\frac{\sqrt{5}}{\pi\sqrt{128\sqrt{2}}}W_t$. In addition, this process has independent increments, as will also be shown in Section 5. These facts, taken together, will prove Theorem 1.

Remark. In [6], Rosen evaluated an integral similar to (2.4). There he also made great use of the fact that the dominant contribution came from regions where $\bigcup_j [s_j, t_j]$ consists of n components of order 2. It seems to this author that this general technique for computing moments would be extremely difficult to apply in the absence of such behavior.

3. Components of order 2

Here we will deal with the aforementioned components of order 2. We will begin by proving

Proposition 1.

$$E[\alpha'_\varepsilon(T)^2] = T\log(1/\varepsilon)^2\left(\frac{10}{128\sqrt{2}\pi^2} + o(1)\right). \tag{3.1}$$

To begin the proof, we write

$$E[\alpha'_\varepsilon(T)^2] = -\frac{1}{(2\pi)^4}\int\int_{D_T^2} \mathrm{e}^{-\varepsilon(p^2+q^2)} E[\mathrm{e}^{\mathrm{i}p(X_{t_1}-X_{s_1})+\mathrm{i}q(X_{t_2}-X_{s_2})}]p_1 q_1 \mathrm{d}^2 s\,\mathrm{d}^2 t\,\mathrm{d}p\,\mathrm{d}q. \tag{3.2}$$

For simplicity we will assume for the time being that $T = 1$. As mentioned in Section 2, in order to handle the expectation in the integrand, we must consider different orderings of the s's and t's. By symmetry, we may assume that $s_1 < s_2$. We will suppress the $\frac{-1}{(2\pi)^4}$ in front of the integral for the time being.

Case 1: $s_1 < s_2 < t_2 < t_1$.

We rewrite the exponent in the expectation as $\mathrm{i}p(X_{t_1} - X_{t_2}) + \mathrm{i}(p+q)(X_{t_2} - X_{s_2}) + \mathrm{i}p(X_{s_2} - X_{s_1})$, and then factor the expectation using independence. As a result, the expectation becomes

$$\mathrm{e}^{-p^2(a+c)-(p+q)^2 b}, \tag{3.3}$$

where $a = t_1 - t_2$, $b = t_2 - s_2$, and $c = s_2 - s_1$. Upon making this linear transformation, the integral in question becomes

$$\int_0^T \left[\int\int\int_{a+b+c \le t_1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2(a+c+\varepsilon)} \mathrm{e}^{-(p+q)^2 b} \mathrm{e}^{-q^2 \varepsilon} p_1 q_1 \mathrm{d}p\mathrm{d}q \right] \mathrm{d}t_1. \tag{3.4}$$

Case 2: $s_1 < s_2 < t_1 < t_2$.

We rewrite the exponent in the expectation as $\mathrm{i}q(X_{t_2} - X_{t_1}) + \mathrm{i}(p+q)(X_{t_1} - X_{s_2}) + \mathrm{i}p(X_{s_2} - X_{s_1})$, and proceed as in Case 1. The integral in question here is then

$$\int_0^T \left[\int\int\int_{a+b+c \le t_1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2(c+\varepsilon)} \mathrm{e}^{-(p+q)^2 b} \mathrm{e}^{-q^2(a+\varepsilon)} p_1 q_1 \mathrm{d}p\mathrm{d}q \right] \mathrm{d}t_1, \tag{3.5}$$

where now $a = t_2 - t_1$, $b = t_1 - s_2$, and $c = s_2 - s_1$.

Case 3: $s_1 < t_1 < s_2 < t_2$.

Here the expectation factors, and since the integrand of (3.2) is then an odd function of p, the contribution to (3.2) of this case is 0.

Remark. Before we begin, let us clear up a technical point that will be necessary later. To do the computations, we will in fact integrate da, db, and dc first. However, if we were to begin with dp and dq, so that the integrals in Cases 1 and 2 were of the form

$$\frac{-1}{(2\pi)^4} \int\int\int\int h(a, b, c, t) \mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t, \tag{3.6}$$

then h would be negative everywhere. Intuitively, this is because when p_1 and q_1 are of the same sign, $|p + q|$ is larger than when they are of the opposite signs. In order to rigorously prove this, just note that the map $\phi((p_1, p_2), (q_1, q_2)) = ((p_1, p_2), (-q_1, q_2))$ is a linear isometry which maps $U = \{p_1 q_1 > 0\}$ bijectively onto $V = \{p_1 q_1 < 0\}$, and the function $|p + q|$ is greater at $(p, q) \in U$ than at $\phi(p, q) \in V$. It will be pointed out later where we have used this fact.

We will attack case 1 first. In order to compute the required integral, we will first examine the following:

$$\int\int\int_{0<a,b,c\le 1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2(a+c+\varepsilon)} \mathrm{e}^{-(p+q)^2 b} \mathrm{e}^{-q^2\varepsilon} p_1 q_1 \mathrm{d}p\mathrm{d}q. \tag{3.7}$$

Upon integrating da, db, and dc, we obtain

$$\begin{aligned} &\int\int \frac{(1-\mathrm{e}^{-p^2})^2}{p^4} \frac{(1-\mathrm{e}^{-(p+q)^2})}{(p+q)^2} \mathrm{e}^{-\varepsilon(p^2+q^2)} p_1 q_1 \mathrm{d}p\mathrm{d}q \\ &= \int\int \frac{(1-\mathrm{e}^{-p^2/\varepsilon})^2}{p^4} \frac{(1-\mathrm{e}^{-(p+q)^2/\varepsilon})}{(p+q)^2} \mathrm{e}^{-(p^2+q^2)} p_1 q_1 \mathrm{d}p\mathrm{d}q. \end{aligned} \tag{3.8}$$

Upon converting to polar coordinates, with $p = r\mathrm{e}^{\mathrm{i}\theta}$, $q = s\mathrm{e}^{\mathrm{i}\phi}$, we arrive at

$$\int\int\int \frac{(1-\mathrm{e}^{-r^2/\varepsilon})^2}{r^2} s^2 \cos(\phi) \mathrm{e}^{-(r^2+s^2)} \int_0^{2\pi} \frac{(1-\mathrm{e}^{-|r\mathrm{e}^{\mathrm{i}\theta}+s\mathrm{e}^{\mathrm{i}\phi}|^2/\varepsilon})}{|r\mathrm{e}^{\mathrm{i}\theta}+s\mathrm{e}^{\mathrm{i}\phi}|^2} \cos(\theta) \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r\mathrm{d}s. \tag{3.9}$$

We will isolate the dθ integral. We may replace θ with $\theta + \phi$. Note that

$$\frac{(1 - \mathrm{e}^{-|r\mathrm{e}^{\mathrm{i}(\theta+\phi)}+s\mathrm{e}^{\mathrm{i}\phi}|^2/\varepsilon})}{|r\mathrm{e}^{\mathrm{i}(\theta+\phi)} + s\mathrm{e}^{\mathrm{i}\phi}|^2} = \int_0^{1/\varepsilon} \mathrm{e}^{-|r\mathrm{e}^{\mathrm{i}(\theta+\phi)}+s\mathrm{e}^{\mathrm{i}\phi}|^2 x}\mathrm{d}x, \tag{3.10}$$

so the dθ integral is

$$\begin{aligned}&\int_0^{1/\varepsilon}\int_0^{2\pi} \mathrm{e}^{-|r\mathrm{e}^{\mathrm{i}(\theta+\phi)}+s\mathrm{e}^{\mathrm{i}\phi}|^2 x}\cos(\theta+\phi)\mathrm{d}\theta\mathrm{d}x \\ &\quad = \int_0^{1/\varepsilon}\int_0^{2\pi} \mathrm{e}^{-|r\mathrm{e}^{\mathrm{i}\theta}+s|^2 x}[\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)]\mathrm{d}\theta\mathrm{d}x. \end{aligned}\tag{3.11}$$

We expand this and pull the $\cos(\phi)$ and $\sin(\phi)$ terms outside of the dθ integral. Integrating the $\sin(\phi)$ term with the $\cos(\phi)$ term already present outside the dθ integral gives 0. Integrating the $\cos(\phi)$ together with the other $\cos(\phi)$ term already present gives $\pi/2$. We have therefore eliminated ϕ from (3.9). The contribution of (3.10) to (3.9) is therefore equal to

$$\frac{\pi}{2}\int_0^{1/\varepsilon} \mathrm{e}^{-(r^2+s^2)x}\int_0^{2\pi} \mathrm{e}^{-2rsx\cos(\theta)}\cos(\theta)\mathrm{d}\theta\mathrm{d}x. \tag{3.12}$$

By [3] (p. 958, 8.431.5 with $\nu = 1$, replace θ with $\theta + \pi$), what remains of the dθ integral is equal to -$2\pi I_1(2rsx)$, where I_1 denotes the modified Bessel function of the first kind. Thus, (3.9) is equal to

$$\begin{aligned}&-\pi^2\int_0^{\infty}\int_0^{\infty}\int_0^{1/\varepsilon} \frac{(1-\mathrm{e}^{-r^2/\varepsilon})^2}{r^2} s^2\mathrm{e}^{-(r^2+s^2)(x+1)} I_1(2rsx)\mathrm{d}x\mathrm{d}r\mathrm{d}s \\ &\quad = -\pi^2\int_0^{\infty}\int_0^{\infty}\int_0^{1/\varepsilon}\int_0^{1/\varepsilon} \mathrm{e}^{-r^2 y}(1-\mathrm{e}^{-r^2/\varepsilon})s^2\mathrm{e}^{-(r^2+s^2)(x+1)} I_1(2rsx)\mathrm{d}y\mathrm{d}x\mathrm{d}r\mathrm{d}s. \end{aligned}\tag{3.13}$$

Expand this into two integrals, and rewrite the first one as

$$-\pi^2\int_0^{\infty}\int_0^{1/\varepsilon}\int_0^{1/\varepsilon} \mathrm{e}^{-s^2(x+1)}s^2\int_0^{\infty} \mathrm{e}^{-r^2(y+x+1)} I_1(2rsx)\mathrm{d}r\mathrm{d}x\mathrm{d}y\mathrm{d}s. \tag{3.14}$$

By [3] (p. 711, 6.618.4) the dr integral is equal to

$$\frac{\sqrt{\pi}}{2\sqrt{x+y+1}}\mathrm{e}^{\frac{(2sx)^2}{8(x+y+1)}} I_{1/2}\left(\frac{(2sx)^2}{8(x+y+1)}\right). \tag{3.15}$$

This is $\frac{(\mathrm{e}^{\frac{s^2x^2}{x+y+1}}-1)}{\sqrt{2}sx}$, since $I_{1/2}(z) = \frac{1}{\sqrt{2\pi z}}(\mathrm{e}^z - \mathrm{e}^{-z})$ by [3] (p. 967, 8.467). Thus, (3.13) is

$$\begin{aligned}&\frac{-\pi^2}{\sqrt{2}}\int_0^{\infty}\int_0^{1/\varepsilon}\int_0^{1/\varepsilon} \mathrm{e}^{-s^2(x+1)} s\frac{(\mathrm{e}^{\frac{s^2x^2}{x+y+1}}-1)}{x}\mathrm{d}x\mathrm{d}y\mathrm{d}s \\ &\quad = \frac{-\pi^2}{\sqrt{2}}\int_0^{1/\varepsilon}\int_0^{1/\varepsilon}\frac{1}{x}\int_0^{\infty} s\left(\mathrm{e}^{-s^2\left(x+1-\frac{x^2}{x+y+1}\right)} - \mathrm{e}^{-s^2(x+1)}\right)\mathrm{d}s\mathrm{d}y\mathrm{d}x. \end{aligned}\tag{3.16}$$

Since

$$\int_0^\infty s\mathrm{e}^{-bs^2}\mathrm{d}s = (1/2)b^{-1}, \tag{3.17}$$

we see that (3.14) is

$$\begin{aligned}
&\frac{-\pi^2}{2\sqrt{2}}\int_0^M\int_0^M \frac{1}{x}\left(\left(x+1-\frac{x^2}{x+y+1}\right)^{-1} - (x+1)^{-1}\right)\mathrm{d}x\mathrm{d}y \\
&\quad = \frac{-\pi^2}{2\sqrt{2}}\int_0^M\int_0^M \frac{x}{(x+1)(xy+2x+y+1)}\mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{3.18}$$

where we have substituted $M = 1/\varepsilon$ to simplify what follows. This last integral is explicitly computable, but it is easier to calculate the derivative and then apply L'Hospital's rule. To do so, we will make use off the following lemma:

Lemma 1. *If $h(x, y, M)$ is bounded, continuous in x and y on $\{x, y \geq 0\}$, differentiable in M with bounded derivative, then*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}M}\int_0^M\int_0^M h(x,y,M)\mathrm{d}x\mathrm{d}y = \int_0^M h(M,y,M)\mathrm{d}y \\
&\quad + \int_0^M h(x,M,M)\mathrm{d}x + \int_0^M\int_0^M \frac{\mathrm{d}}{\mathrm{d}M}h(x,y,M)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.19}$$

In the case of (3.18) the integrand does not depend on M, so the last term is 0. We include the last term because it will be used later. By the lemma, the derivative with respect to M of (3.18) is

$$\begin{aligned}
&\frac{-\pi^2}{2\sqrt{2}}\left[\frac{M}{M+1}\int_0^M \frac{\mathrm{d}y}{(M+1)y+2M+1} + \int_0^M \frac{x}{(x+1)((M+2)x+M+1)}\mathrm{d}x\right] \\
&\quad = \frac{-\pi^2}{2\sqrt{2}}\left[\frac{M}{M+1}\int_0^M \frac{\mathrm{d}y}{(M+1)y+2M+1}\right. \\
&\qquad \left. + \int_0^M \frac{1}{(x+1)} - \frac{M+1}{((M+2)x+M+1)}\mathrm{d}x\right].
\end{aligned} \tag{3.20}$$

Performing the integration gives

$$\begin{aligned}
&\frac{-\pi^2}{2\sqrt{2}}\left[\frac{M}{(M+1)^2}\log\left(\frac{(M+1)M+2M+1}{2M+1}\right) + \log(M+1)\right. \\
&\quad \left. - \frac{(M+1)}{(M+2)}\log\left(\frac{(M+2)M+M+1}{M+1}\right)\right].
\end{aligned} \tag{3.21}$$

The expression inside the brackets is asymptotic to $\frac{2\log M}{M}$. This is an immediate consequence of the following easily verified facts:

(1) $\frac{M}{(M+1)^2} = \frac{1}{M} + O(\frac{1}{M^2})$.

(2) $\log(\frac{(M+1)M+2M+1}{2M+1}) = \log M + O(1)$.

(3) $\log(M+1) = \log(M) + O(\frac{1}{M})$.

(4) $\log(\frac{(M+2)M+M+1}{M+1}) = \log M + O(\frac{1}{M})$.
(5) $\frac{(M+1)}{(M+2)} = 1 - \frac{1}{(M+2)} = 1 - \frac{1}{M} + O(\frac{1}{M^2})$.

Thus, (3.20) is equal to $\frac{\log M}{M}(\frac{-\pi^2}{\sqrt{2}} + o(1))$, and it follows from L'Hospital's rule that (3.18) is equal to $(\log M)^2(\frac{-\pi^2}{2\sqrt{2}} + o(1))$. Recall that we split (3.13) into two integrals. We must now deal with the second, namely

$$-\pi^2 \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} \mathrm{e}^{-s^2(x+1)} s^2 \int_0^\infty \mathrm{e}^{-r^2(y+x+1+1/\varepsilon)} I_1(2rsx) \mathrm{d}r \mathrm{d}x \mathrm{d}y \mathrm{d}s. \tag{3.22}$$

We can follow steps (3.14)–(3.18) exactly, with the only difference being that we have $y + M$ in place of y. We get

$$\begin{aligned} &\frac{-\pi^2}{2\sqrt{2}} \int_0^M \int_0^M \frac{1}{x} \left(\left(x + 1 - \frac{x^2}{x+y+1+M} \right)^{-1} - (x+1)^{-1} \right) \mathrm{d}x \mathrm{d}y \\ &\quad = \frac{-\pi^2}{2\sqrt{2}} \int_0^M \int_0^M \frac{x}{(x+1)(xy+2x+y+1+M(x+1))} \mathrm{d}x \mathrm{d}y, \end{aligned} \tag{3.23}$$

where, again, $M = 1/\varepsilon$. We take the derivative as before, using Lemma 1, and this time the integrand depends on M:

$$\begin{aligned} &\frac{-\pi^2}{2\sqrt{2}} \left[\frac{M}{M+1} \int_0^M \frac{\mathrm{d}y}{(M+1)y + 2M + 1 + M(M+1)} \right. \\ &\quad + \int_0^M \frac{x}{(x+1)((2M+2)x+2M+1)} \mathrm{d}x \\ &\quad \left. - \int_0^M \int_0^M \frac{x}{(xy+2x+y+1+M(x+1))^2} \mathrm{d}x \mathrm{d}y \right]. \end{aligned} \tag{3.24}$$

The first term,

$$\frac{M}{M+1} \int_0^M \frac{\mathrm{d}y}{(M+1)y + 2M + 1 + M(M+1)}, \tag{3.25}$$

is bounded above by

$$\int_0^M \frac{\mathrm{d}y}{M^2} = \frac{1}{M}. \tag{3.26}$$

We may ignore it, as it is $o(\log M/M)$. The second term,

$$\int_0^M \frac{x}{(x+1)((2M+2)x+2M+1)} \mathrm{d}x = \int_0^M \frac{1}{(x+1)} - \frac{2M+1}{((2M+2)x+2M+1)} \mathrm{d}x, \tag{3.27}$$

is the same as the second integral in (3.20), with $2M$ replacing M. We can follow steps (3.20) and (3.21), and use fact (3) above along with
(6) $\log(\frac{(2M+2)M+2M+1}{M+1}) = \log M + O(\frac{1}{M})$,
(7) $\frac{(M+1)}{(M+2)} = 1 - \frac{1}{(2M+2)} = 1 - \frac{1}{2M} + O(\frac{1}{M^2})$,
to see that this term is asymptotic to $(1/2)\log(M)/M$. The third term,

$$\int_0^M \int_0^M \frac{x}{(xy + 2x + y + 1 + M(x + 1))^2} \mathrm{d}x\mathrm{d}y, \tag{3.28}$$

is bounded above by

$$\int_0^M \int_0^M \frac{\mathrm{d}x\mathrm{d}y}{M^2(x + 1)^2} = O(1/M) \tag{3.29}$$

and may also be ignored. Thus, (3.22) is $(\log(1/\varepsilon))^2(\frac{-\pi^2}{8\sqrt{2}}+o(1))$. We have found the asymptotics for (3.22) and (3.14). They are

$$\begin{aligned} &-\pi^2 \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} \mathrm{e}^{-s^2(x+1)} s^2 \int_0^\infty \mathrm{e}^{-r^2(y+x+1)} I_1(2rsx)\mathrm{d}r\mathrm{d}x\mathrm{d}y\mathrm{d}s \\ &\quad = (\log(1/\varepsilon))^2 \left(\frac{-\pi^2}{2\sqrt{2}} + o(1)\right), \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} &-\pi^2 \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} \mathrm{e}^{-s^2(x+1)} s^2 \int_0^\infty \mathrm{e}^{-r^2(y+x+1+1/\varepsilon)} I_1(2rsx)\mathrm{d}r\mathrm{d}x\mathrm{d}y\mathrm{d}s \\ &\quad = (\log(1/\varepsilon))^2 \left(\frac{-\pi^2}{8\sqrt{2}} + o(1)\right). \end{aligned} \tag{3.31}$$

Combining these as in (3.13), we see that (3.7) is $\log(1/\varepsilon)^2(\frac{-3\pi^2}{8\sqrt{2}} + o(1))$.

Now let us consider

$$\int\int\int_{0<a,b,c\le t} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2(a+c+\varepsilon)} \mathrm{e}^{-(p+q)^2 b} \mathrm{e}^{-q^2\varepsilon} p_1 q_1 \mathrm{d}p\mathrm{d}q. \tag{3.32}$$

By a simple scaling we now show that this is equal to $(\log(t/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}} + o(1))$. The scaling is as follows:

$$\begin{aligned} &\int\int\int_{0<a,b,c\le t} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2(a+c+\varepsilon)} \mathrm{e}^{-(p+q)^2 b} \mathrm{e}^{-q^2\varepsilon} p_1 q_1 \mathrm{d}p\mathrm{d}q \\ &\quad = \int\int\int_{0<a,b,c\le t} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2 t(\frac{a+c+\varepsilon}{t})} \mathrm{e}^{-(p+q)^2 t\frac{b}{t}} \mathrm{e}^{-q^2 t(\frac{\varepsilon}{t})} p_1 q_1 \mathrm{d}p\mathrm{d}q \\ &\quad = t^2 \int\int\int_{0<a,b,c\le 1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &\qquad \times \int\int \mathrm{e}^{-p^2 t(a+c+\varepsilon/t)} \mathrm{e}^{-(p+q)^2 tb} \mathrm{e}^{-q^2 t(\varepsilon/t)} (\sqrt{t}p_1)(\sqrt{t}q_1)\mathrm{d}p\mathrm{d}q. \end{aligned} \tag{3.33}$$

Now replace (p, q) with $(p/\sqrt{t}, q/\sqrt{t})$. The t^2 in front of the integral is canceled, and we are left with

$$\begin{aligned} &\int\int\int_{0<a,b,c\le 1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \int\int \mathrm{e}^{-p^2(a+c+\varepsilon/t)} \mathrm{e}^{-(p+q)^2 b} \mathrm{e}^{-q^2(\varepsilon/t)} p_1 q_1 \mathrm{d}p\mathrm{d}q \\ &\quad = (\log(t/\varepsilon))^2 \left(\frac{-3\pi^2}{8\sqrt{2}} + o(1)\right). \end{aligned} \tag{3.34}$$

We must now examine the same integral, but over the region $\{a+b+c < t_1\}$ rather than $\{0 < a, b, c \le t_1\}$. However, the remark following Case 3 shows that if $U \subseteq V$, then

$$\left|\int_{a,b,c\in U}\right| \le \left|\int_{a,b,c\in V}\right|. \tag{3.35}$$

We also note that

$$\frac{(\log(t/\varepsilon))^2}{(\log(1/\varepsilon))^2} \longrightarrow 1. \tag{3.36}$$

We can then write

$$\left|\int_{a,b,c<(t_1/3)}\right| \le \left|\int_{a+b+c<t_1}\right| \le \left|\int_{a,b,c<t_1}\right|. \tag{3.37}$$

The first and last integrals are both $(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$, and it follows that the middle one is as well. Thus, the integrand in the $\mathrm{d}t_1$ integral (3.4) is $(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$, with the $o(1)$ term uniformly bounded on $\{0<\delta<t_1<T\}$. We will split up (3.4) as:

$$\begin{aligned}&\int_0^T\int\int\int_{a+b+c\le t_1}\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t_1\int\int \mathrm{e}^{-p^2(a+c+\varepsilon)}\mathrm{e}^{-(p+q)^2b}\mathrm{e}^{-q^2\varepsilon}p_1q_1\mathrm{d}p\mathrm{d}q\\ &\quad=\int_0^\delta h(t_1,\varepsilon)\mathrm{d}t_1+\int_\delta^T h(t_1,\varepsilon)\mathrm{d}t_1,\end{aligned} \tag{3.38}$$

where h denotes the result after doing the integrals in the other variables. We know that, for the second integral, $h(t_1,\varepsilon)(\log(1/\varepsilon))^{-2} \longrightarrow \frac{-3\pi^2}{8\sqrt{2}}$ uniformly. Thus, the second integral is $(T-\delta)(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$. The first integral is bounded above in absolute value by (assuming that $\delta<1$)

$$\begin{aligned}&\left|\int_0^\delta\int\int\int_{a+b+c\le 1}\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t_1\int\int \mathrm{e}^{-p^2(a+c+\varepsilon)}\mathrm{e}^{-(p+q)^2b}\mathrm{e}^{-q^2\varepsilon}p_1q_1\mathrm{d}p\mathrm{d}q\right|\\ &\quad=\delta(\log(1/\varepsilon))^2\left(\frac{3\pi^2}{8\sqrt{2}}+o(1)\right).\end{aligned} \tag{3.39}$$

By letting $\delta \longrightarrow 0$, we may finally conclude that the contribution from Case 1 is

$$T(\log(1/\varepsilon))^2\left(\frac{-3\pi^2}{8\sqrt{2}}+o(1)\right). \tag{3.40}$$

Similar techniques will yield Case 2. We will give only the outline here. Recall that we are evaluating

$$\int_0^T\int\int\int_{a+b+c\le t_1}\mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}t_1\int\int \mathrm{e}^{-p^2(c+\varepsilon)}\mathrm{e}^{-(p+q)^2b}\mathrm{e}^{-q^2(a+\varepsilon)}p_1q_1\mathrm{d}p\mathrm{d}q. \tag{3.41}$$

As before, we begin by changing the domain to $\{0 \leq a, b, c \leq 1\}$ and integrating da, db, and dc. We arrive at

$$\int\int \frac{(1-\mathrm{e}^{-p^2})}{p^2}\frac{(1-\mathrm{e}^{-q^2})}{q^2}\frac{(1-\mathrm{e}^{-(p+q)^2})}{(p+q)^2}\mathrm{e}^{-\varepsilon(p^2+q^2)}p_1q_1\mathrm{d}p\mathrm{d}q$$
$$= \int\int \frac{(1-\mathrm{e}^{-p^2/\varepsilon})}{p^2}\frac{(1-\mathrm{e}^{-q^2/\varepsilon})}{q^2}\frac{(1-\mathrm{e}^{-(p+q)^2/\varepsilon})}{(p+q)^2}\mathrm{e}^{-(p^2+q^2)}p_1q_1\mathrm{d}p\mathrm{d}q. \tag{3.42}$$

We convert to polar coordinates again:

$$\int\int\int (1-\mathrm{e}^{-r^2/\varepsilon})(1-\mathrm{e}^{-s^2/\varepsilon})\cos(\phi)\mathrm{e}^{-(r^2+s^2)}$$
$$\times \int_0^{2\pi} \frac{(1-\mathrm{e}^{-|r\mathrm{e}^{\mathrm{i}\theta}+s\mathrm{e}^{\mathrm{i}\phi}|^2/\varepsilon})}{|r\mathrm{e}^{\mathrm{i}\theta}+s\mathrm{e}^{\mathrm{i}\phi}|^2}\cos(\theta)\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r\mathrm{d}s. \tag{3.43}$$

We follow the same steps for the dθ integral as before (steps (3.9)–(3.13)) to arrive at

$$-\pi^2\int_0^\infty\int_0^\infty\int_0^{1/\varepsilon}(1-\mathrm{e}^{-r^2/\varepsilon})(1-\mathrm{e}^{-s^2/\varepsilon})\mathrm{e}^{-(r^2+s^2)(x+1)}I_1(2rsx)\mathrm{d}x\mathrm{d}r\mathrm{d}s. \tag{3.44}$$

We will expand this into 4 integrals and do each separately. The first is

$$-\pi^2\int_0^\infty\int_0^{1/\varepsilon}\mathrm{e}^{-s^2(x+1)}\left(\int_0^\infty \mathrm{e}^{-r^2(x+1)}I_1(2rsx)\mathrm{d}r\right)\mathrm{d}x\mathrm{d}s. \tag{3.45}$$

The dr integral, as in step (3.15), is equal to $\frac{(\mathrm{e}^{\frac{s^2x^2}{x+1}}-1)}{\sqrt{2}sx}$, and we obtain:

$$\frac{-\pi^2}{\sqrt{2}}\int_0^\infty\int_0^{1/\varepsilon}\mathrm{e}^{-s^2(x+1)}\frac{(\mathrm{e}^{\frac{s^2x^2}{x+1}}-1)}{sx}\mathrm{d}x\mathrm{d}s. \tag{3.46}$$

We use the identity

$$s\int_0^{\frac{x^2}{x+1}}\mathrm{e}^{s^2t}\mathrm{d}t = \frac{(\mathrm{e}^{\frac{s^2x^2}{x+1}}-1)}{s}, \tag{3.47}$$

and the integral in question becomes

$$\frac{-\pi^2}{\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}\int_0^{\frac{x^2}{x+1}}\int_0^\infty \mathrm{e}^{-s^2(x+1-t)}s\mathrm{d}s\mathrm{d}t\mathrm{d}x. \tag{3.48}$$

Note that $t \leq \frac{x^2}{x+1} < x+1$, which implies $(x+1-t) > 0$, so there is no problem with convergence. Tackling this integral again reduces to basic calculus. Begin by substituting $u = s^2(x+1-t)$ to get

$$\frac{-\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}\int_0^{\frac{x^2}{x+1}}\frac{1}{x+1-t}\int_0^\infty \mathrm{e}^{-u}\mathrm{d}u\mathrm{d}t\mathrm{d}x = \frac{-\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}\int_0^{\frac{x^2}{x+1}}\frac{1}{x+1-t}\mathrm{d}t\mathrm{d}x$$
$$= \frac{-\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}\log\left(\frac{x+1}{x+1-x^2/(x+1)}\right)\mathrm{d}x. \tag{3.49}$$

If $M = 1/\varepsilon$, then, by the fundamental theorem of calculus, $\frac{\mathrm{d}}{\mathrm{d}M}$ of the above is

$$\frac{-\pi^2}{2\sqrt{2}} \frac{\log\left(\frac{M+1}{M+1-M^2/(M+1)}\right)}{M}. \tag{3.50}$$

Since

$$\log\left(\frac{M+1}{M+1-M^2/(M+1)}\right) = \log(M) + O(1), \tag{3.51}$$

we see that (3.50) is $\frac{\log M}{M}(\frac{-\pi^2}{2\sqrt{2}} + o(1))$, and thus our original integral (3.45) is $(\log M)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$. Recall that (3.44) was divided into four integrals. The remaining three give a contribution of

$$-\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} (-\mathrm{e}^{-r^2/\varepsilon} - \mathrm{e}^{-s^2/\varepsilon} + \mathrm{e}^{-r^2/\varepsilon}\mathrm{e}^{-s^2/\varepsilon})\mathrm{e}^{-(r^2+s^2)(x+1)} I_1(2rsx)\mathrm{d}x\mathrm{d}r\mathrm{d}s. \tag{3.52}$$

The integral corresponding to the first and second term will be identical, and so if we follow steps (3.45) through (3.49) we get

$$\begin{aligned} &\frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log\left(\frac{x+1+1/\varepsilon}{x+1+1/\varepsilon - x^2/(x+1+1/\varepsilon)}\right) \mathrm{d}x \\ &\quad + 2\frac{\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log\left(\frac{x+1+1/\varepsilon}{x+1+1/\varepsilon - x^2/(x+1)}\right) \mathrm{d}x \\ &= \frac{\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log\left(1 - \frac{x^2}{(x+1+1/\varepsilon)^2}\right) \mathrm{d}x \\ &\quad - 2\frac{\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log\left(1 - \frac{x^2}{(x+1)(x+1+1/\varepsilon)}\right) \mathrm{d}x. \end{aligned} \tag{3.53}$$

Each of these integrals is $O(\log(1/\varepsilon))$. To see this, note that the absolute values of the integrands are bounded above by $g(x) = \frac{-1}{x}\log(1-x^2)$ on $\{0 < x < \frac{1}{2}\}$. $g(x)$ is bounded on this interval, so the integral over $\{0 < x < \frac{1}{2}\}$ is bounded. For $\{x > \frac{1}{2}\}$ the integrand is bounded by $\frac{-1}{x}\log(\frac{1}{2})$, since $1/\varepsilon > x$, and it follows from this that the integral is $O(\log(1/\varepsilon))$. This proves that (3.43) is $\log(1/\varepsilon)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$. The remainder of the proof that the integral we began with in case 2, namely (3.5), is $\log(1/\varepsilon)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$ is identical to the steps (3.35) through (3.40).

Combining our work in Cases 1 and 2, and reinserting the constant $\frac{-1}{(2\pi)^4}$ which was suppressed throughout, we see that

$$E[\alpha'_\varepsilon(T)^2] = T\log(1/\varepsilon)^2\left(\frac{5}{128\sqrt{2}\pi^2} + o(1)\right). \tag{3.54}$$

We assumed at the outset that $s_1 < s_2$, so this must be multiplied by 2 to obtain the correct answer. □

Corollary 1. *The contribution to* (2.4) *of any component of order 2 is* $\log(1/\varepsilon)^2(\frac{5}{128\sqrt{2}\pi^2}+o(1))$. *That is, if* r_j *is the left endpoint of a component of order 2, and* r_{j+1} *is the maximal right endpoint (see* (2.7)*), then the integral over this region is* $\log(1/\varepsilon)^2(\frac{5}{128\sqrt{2}\pi^2}+o(1))(r_{j+1}-r_j)$.

Proof. Suppose that the component of order 2 is composed of $[s_k, t_k]$ and $[s_{k'}, t_{k'}]$. Then the integral in question is

$$\frac{-1}{(2\pi)^4}\int\int_{s_k,t_k,s_{k'},t_{k'}\in[r_j,r_{j+1}]} \mathrm{e}^{-\varepsilon(p^2+q^2)}E[\mathrm{e}^{\mathrm{i}p(X_{t_k}-X_{s_k})+iq(X_{t_{k'}}-X_{s_{k'}})}]p_1q_1\mathrm{d}^2s\mathrm{d}^2t\mathrm{d}p\mathrm{d}q. \tag{3.55}$$

We can write $X_{t_k}-X_{s_k}$ as $(X_{t_k}-X_{r_j})-(X_{s_k}-X_{r_j})$, and likewise for $X_{t_{k'}}-X_{s_{k'}}$. Since $X_{r_j+t}-X_{r_j}$ is itself a Brownian motion we see that (3.55) is equal to (3.2) with D_T replaced by $D_{r_{j+1}-r_j}$. □

4. Components of order $n \geq 3$

We now turn to components of order n, where $n \geq 3$. We will show

Proposition 2. *The contribution to* (2.4) *of any component of order* $n \geq 3$ *is* $o(\log(1/\varepsilon)^n)$.

This entire section is devoted to the proof of this proposition. Suppose that a component of order n is can be formed by a specific arrangement of intervals $[s_1, t_1], \ldots, [s_n, t_n]$ corresponding to the variables $p_1, \ldots, p_n$. Let $\{r_1, \ldots, r_{2n}\}$ be a relabeling of the s_i's and t_i's so that $r_1 \leq r_2 \leq \cdots \leq r_{2n}$. We split up the expectation in the integrand by independence and change the coordinates as we did in the beginning of Section 3. The contribution of this arrangement of intervals is then given by the integral

$$\int \mathrm{e}^{-\varepsilon\sum p_i^2}\prod_i (p_i)_1\left(\int_{\sum c_j<T}\prod_j \mathrm{e}^{-u_j^2c_j}\prod_j \mathrm{d}c_j\right)\prod_i \mathrm{d}p_i, \tag{4.1}$$

where $c_j = r_{j+1}-r_j$ and, as before, $(p_i)_1$ denotes the x-coordinate of p_i. Each u_j is a linear combination of p_i's corresponding to the interval $[r_{j-1}, r_j]$. There is a nice visual device, shown to me by Jay Rosen, which allows one to more easily visualize the relationship between the u_j's and p_i's. We draw an arc corresponding to each p_i and whose endpoints are in the correct order. The intervals between endpoints belong to u_j's, and each u_j is the linear combination of the p_i's which arch over it. As an example, let us consider the order 4 component with $s_1 < s_2 < s_3 < t_1 < t_2 < s_4 < t_3 < t_4$. The associated picture to this configuration is

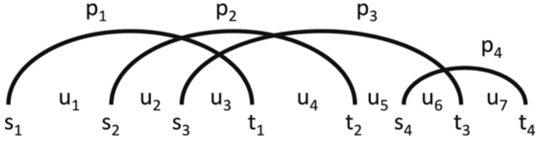

Fig. 1. Sample configuration.

From this picture we see easily that $u_1 = p_1, u_2 = p_1+p_2, u_3 = p_1+p_2+p_3, u_4 = p_2+p_3$, $u_5 = p_3, u_6 = p_3+p_4$, and $u_7 = p_4$. A reader confused by what follows in this paper is advised to work a few special cases, e.g. components of order 3, using this visual aid.

For each j, either $u_j - u_{j-1} = p_i$ or $u_j - u_{j-1} = -p_i$ for some i. In the first case we will refer to u_j as increasing (abbreviated as $u_j \uparrow$) and in the second case we will say u_j is decreasing (abbreviated as $u_j \downarrow$). In the example above, we have $u_1, u_2, u_3, u_6 \uparrow$ and $u_4, u_5, u_7 \downarrow$. In other words, referring to Fig. 1, u intervals are increasing if they lie directly to the right of s values, and decreasing if they are directly to the right of t values. To simplify some of the notations that follows, let $u_0 = u_{2n} = 0$. Let us define an interval $[s_i, t_i]$ to be *isolated* if $t_k, s_k \notin (s_i, t_i)$ for all k. If $[s_i, t_i]$ is isolated, we may abuse the notation where convenient and refer to p_i or the u_j containing p_i as isolated as well. For example, in Fig. 2, we could say that p_3, $[s_3, t_3]$, or u_4 are isolated.

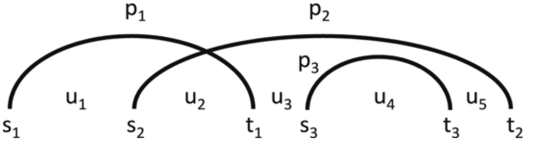

Fig. 2. u_4 is isolated.

It turns out that isolated intervals such as u_4 are a bit troublesome to deal with, so let us first suppose that we are dealing with a configuration which contains no isolated intervals, such as Fig. 1. In this case, it will be shown that we may obtain a sufficient bound for (4.1) by replacing the integrand with the absolute value of the integrand. We may then also replace the region $\sum c_j < T$ with $0 < c_j < T$ for all j and use the simple fact that

$$\int_0^T \mathrm{e}^{-u_j^2 c_j} \mathrm{d}c_j \le \frac{k}{(1+|u_j|)^2} \tag{4.2}$$

for a constant k to reduce our problem to bounding

$$\int \frac{\mathrm{e}^{-\varepsilon \sum p_i^2} \prod |p_i|}{\prod_{j=1}^{2n-1} (1+|u_j|)^2} \prod \mathrm{d}p_i. \tag{4.3}$$

We will eventually need the following lemmas:

Lemma 2.

$$\int_{R^2} \frac{\mathrm{e}^{-\varepsilon u^2}}{(1+|u|)^2} \mathrm{d}^2 u = O(\log(1/\varepsilon)). \tag{4.4}$$

Proof. Replace u with $u/\sqrt{\varepsilon}$ and the integral becomes

$$\int_{R^2} \frac{\mathrm{e}^{-u^2}}{(\sqrt{\varepsilon}+u)^2} \mathrm{d}^2 u. \tag{4.5}$$

The integral over $|u| > 1$ is clearly $O(1)$, and for $|u| < 1$ it suffices to bound

$$\int_{|u|<1} \frac{1}{(\sqrt{\varepsilon}+u)^2} \mathrm{d}^2 u. \tag{4.6}$$

Converting to polar coordinates this is

$$2\pi \int_0^1 \frac{r}{(\sqrt{\varepsilon}+r)^2} \mathrm{d}r \le 2\pi \int_0^1 \frac{1}{\sqrt{\varepsilon}+r} \mathrm{d}r = O(\log(1/\varepsilon)). \quad \square \tag{4.7}$$

Lemma 3. *There is a constant $c < \infty$ such that, independently of $\varepsilon > 0, a \in R^2$ and $n \geq 1$, we have*

$$\int_{R^2} \frac{e^{-\varepsilon u^2}}{(1+|u|)^2(1+|a+u|)^n} d^2u < c. \tag{4.8}$$

Proof. We can ignore the numerator. By Hölder's inequality the integral is bounded by

$$\left(\int \frac{1}{(1+|u|)^3} d^2u\right)^{2/3} \left(\int \frac{1}{(1+|u+a|)^{3n}} d^2u\right)^{1/3} < c < \infty. \quad \square \tag{4.9}$$

These lemmas motivate the intuition behind our approach, which we first describe informally. The lemmas essentially say that a square in the denominator gives a log, whereas a cube or higher gives a constant. We will write (4.3) in such a way that we can cancel the $\prod |p_i|$ in the numerator with powers in the denominator. We will use the Cauchy–Schwarz inequality to cut down on the number of different terms in the numerator, and we will change the variables by a linear transformation. After all of this we will obtain a product of a collection of integrals of the form in Lemma 2 with at least one integral in the form of Lemma 3. Each one of the type in Lemma 2 contributes a $\log(1/\varepsilon)$, whereas the Lemma 3 type does not. When we multiply everything out, the power of $\log(1/\varepsilon)$ will be less than n. (As a side note, Lemma 2 also indicates why we are initially not considering the case of the isolated intervals. In that case there is some p_i which is only present as a term in one u_j, so that if we were to put the absolute value inside the integral as we are doing here, we would have only a square of p_i in the denominator with a $|p_i|$ present in the numerator. This would essentially give us

$$\int \frac{e^{-\varepsilon p_i^2}}{1+|p_i|} d^2 p_i. \tag{4.10}$$

This is only $O(\sqrt{1/\varepsilon})$, which is not good enough.)

To make this approach good, we will need a way to make sure that, after we cancel the terms in the numerator, we have enough terms left in the denominator to obtain adequate convergence. In terms of the sheer number of powers in the denominator there is no problem. Lemma 2 suggests we need more than $2n$ powers on the bottom, but there are n powers on top versus $4n-2$ on the bottom, for a total of $3n-2$ on the bottom. This is enough since $n \geq 3$. The tricky part is making sure that we have a proper assortment on the bottom. The details are rather involved, so we first will prove several technical lemmas. To state the first lemma, we need another bit of terminology. We will say that p_i is *t-free* if there is no t_k contained in (s_i, t_i). For example, in Fig. 3 p_2 is t-free, but no others are.

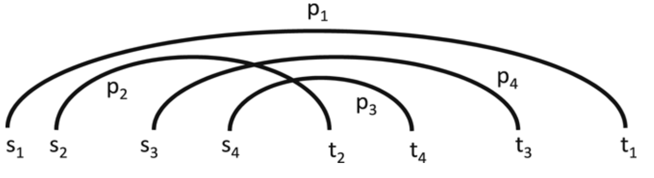

Fig. 3. p_2 is t-free.

Another way to characterize the t-free p_i's is as all p_i elements which are contained only as terms in increasing u_j's. The relationship between decreasing u_j's and t-free p_i's is critical for our purposes, and we have the following lemma, which was proved in [7].

Lemma 4. *The span of the decreasing u_j's is equal to the span of the set of all p_i's which are not t-free. Furthermore, suppose that for each t-free p_i we choose $u(p_i)$ to be any one of the increasing u_j's which contains p_i as a term. Then, if we let $D = \{$set of decreasing u_j's$\}$ $\bigcup\{$set of all $u(p_i)$'s$\}$, D spans the entire set $\{p_1, \dots, p_n\}$.*

Proof. To begin with, suppose that p_i is t-free. Then p_i only appears as a term in increasing u_j's, and so is not in the span of the decreasing u_j's. Conversely, suppose there exist non-t-free p's which are not contained in the span of the decreasing u_j's. Let p_{i_o} be the non-t-free p with largest s value which is not in the span of the decreasing u_j's. That is, if $s_i > s_{i_o}$ and p_i is non-t-free, then p_i is in the span of decreasing u_j's. Now, let u_{j_o} be the u with largest j value which contains p_{i_o}, i.e. u_{j_o} satisfies $u_{j_o+1} - u_{j_o} = -p_{i_o}$. Then u_{j_o+1} is necessarily decreasing, and if we can show that u_{j_o} is in the span of decreasing u_j's it will follow that p_{i_o} is as well, a contradiction. Assume to the contrary, that u_{j_o} is increasing. Let $v > 0$ be chosen as small as possible so that u_{j_o-v} is decreasing. The fact that p_{i_o} is not t-free implies that p_{i_o} appears as a term in u_{j_o-v}. Furthermore, we can write $p_{i_o} = u_{j_o-v} + (u_{j_o-v+1} - u_{j_o-v}) + \cdots + (u_{j_o} - u_{j_o-1}) - u_{j_o+1}$. Now, u_{j_o-v} and u_{j_o+1} are decreasing, and the terms $(u_{j_o-v+1} - u_{j_o-v}), \dots, (u_{j_o} - u_{j_o-1})$ are each equal to some $p_{i'}$ which has the properties that (i) $p_{i'}$ is not t-free, because it appears as a term in u_{j_o+1}, and (ii) $p_{i'}$ has larger s values than p_{i_o}. We conclude that every such $p_{i'}$ is in the span of the decreasing u_j's, which means that p_{i_o} is as well. This is a contradiction, and establishes the first part of the lemma. To prove the second part, just note that $u(p_i)$ contains p_i as a term as well as several other p_k's which cannot be t-free. These p_k's are in the span of D then, and thus p_i is as well. □

In order to state the next lemma, we must consider (4.3) again. Let $u_{j_1}, \dots, u_{j_n}$ be the increasing u's in order. That is, $u_{j_i} - u_{j_i-1} = p_i$. We see that (4.3) is bounded by

$$\int \frac{e^{-\varepsilon \sum p_i^2} \prod(|u_{j_i}| + |u_{j_i-1}|)}{\prod_{j=1}^{2n-1} (1 + |u_j|)^2} \prod \mathrm{d}p_i. \tag{4.11}$$

Expand the numerator completely, and break this integral into the sum of many integrals, each of which we do individually. Each of these integrals has a product of $|u|$'s in the numerator, but no u can appear more than twice. This allows us to cancel all of the u's in the numerator with u's in the denominator (Note: The word "canceling", in this context, means replacing $\frac{|u|}{1+|u|}$ with 1.). We arrive at the following integral:

$$\int \frac{e^{-\varepsilon \sum p_i^2}}{\prod_{j=1}^{2n-1} (1 + |u_j|)^{m_j}} \prod \mathrm{d}p_i, \tag{4.12}$$

where $m_j = 0$, 1, or 2, depending on what power of u_j appeared in the numerator. The following lemma relates m_j with the properties of u_j in the configuration of intervals.

Lemma 5. (1) *If* $u_j \downarrow$ *then* $m_j, m_{j-1} \geq 1$.
(2) *If* $u_j \downarrow$ *and* $m_j = 1$, *then* $u_{j+1} \uparrow$ *and* $m_{j+1} \geq 1$.
(3) *If* $u_j, u_{j+1} \downarrow$ *then* $m_j = 2$.

Proof. Each term in the numerator is of the form $(|u_j| + |u_{j-1}|)$ where $u_j \uparrow$. We see that we can only have $m_j = 0$ if u_j appears in two terms in the numerator, and this can only happen if both u_j and u_{j+1} are increasing. This proves (1). If $u_j \downarrow$ then u_j appears at most once in the numerator in the term $(|u_{j+1}| + |u_j|)$ where u_{j+1} must be increasing. Furthermore, u_{j+1} can appear in at most one other term, and so if $m_j = 1$ then $m_{j+1} \geq 1$. This proves (2). As for (3), if $u_j, u_{j+1} \downarrow$ then u_j does not appear in the numerator at all, so $m_j = 2$. □

We now turn our attention to (4.3). Suppose that we can form sets $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_s\}$ with the following properties:

(i) Each of $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ are equal to some u_j.
(ii) A and B each span $\{p_1, \ldots, p_n\}$.
(iii) If $a_i = u_j$ or $b_i = u_j$, then $m_j \geq 1$.
(iv) If $a_i = b_k = u_j$, then $m_j = 2$.

Note that if we can find such sets we can, simply by deleting elements if necessary, find two sets A_o and B_o which satisfy the above properties and each of which have n elements. So in the calculations which follow we will assume that $n = s = r$, even though when we eventually construct A and B they may have more than n elements. Given two such sets, we could bound (4.12) by

$$\int \frac{e^{-\varepsilon \sum p_i^2}}{K(a_1, \ldots, a_n) \prod_{j=1}^{n} (1 + |a_j|)(1 + |b_j|)} \prod \mathrm{d}p_i, \tag{4.13}$$

where K is of the form $(1 + |u_j|)$ for some j. Recall that we have $3n - 2$ powers of u's in the denominator, so there will always be at least one term left over after choosing our sets A and B. This term will contain a linear combination of p_i's, but since A spans $\{p_1, \ldots, p_n\}$ we may write it as a linear combination of a_j's. It is irrelevant what the linear combination present in K actually is, except that it must be nontrivial. Now, we can apply the Cauchy–Schwarz inequality to bound (4.13) by

$$\left(\int \frac{e^{-\varepsilon \sum p_i^2}}{K(a_1, \ldots, a_n)^2 \prod_{j=1}^{n} (1 + |a_j|)^2} \prod \mathrm{d}p_i \right)^{1/2} \times \left(\int \frac{e^{-\varepsilon \sum p_i^2}}{\prod_{j=1}^{n} (1 + |b_j|)^2} \prod \mathrm{d}p_i \right)^{1/2}. \tag{4.14}$$

There is a constant $c > 0$ so that $\sum p_i^2 > c \sum a_i^2$ and $\sum p_i^2 > c \sum b_i^2$; this is because the functions $\sum a_i^2$ and $\sum b_i^2$ are homogeneous of degree 2 in the p_i's and bounded on $\sum p_i^2 = 1$. Thus, (4.14) is bounded by

$$\left(\int \frac{e^{-\varepsilon c \sum a_j^2}}{K(a_1, \ldots, a_n)^2 \prod_{j=1}^{n} (1 + |a_j|)^2} \prod \mathrm{d}p_i \right)^{1/2} \times \left(\int \frac{e^{-\varepsilon c \sum b_j^2}}{\prod_{j=1}^{n} (1 + |b_j|)^2} \prod \mathrm{d}p_i \right)^{1/2}. \tag{4.15}$$

Now, we apply a linear change of coordinates to these integrals so that we are integrating with respect to a_j and b_j instead of p_i. Relabel if necessary so that a_1 is one of the a's which appears as a term in $K(a_1, \dots, a_n)$. We see that the first integral in (4.15) is bounded by a constant times

$$\int \left(\int \frac{e^{-c\varepsilon a_1^2}}{K(a_1, \dots, a_n)^2 (1 + |a_1|)^2} da_1 \right) \prod_{j=2}^{n} \frac{e^{-c\varepsilon a_j^2}}{(1 + |a_j|)^2} da_j. \tag{4.16}$$

By Lemma 3 the inner integral is $O(1)$ and by Lemma 2 the others are all $O(\log(1/\varepsilon))$. Lemma 2 also shows that the second integral in (4.15) is $O(\log^n(1/\varepsilon))$. We see that (4.15) is $O((\log(1/\varepsilon))^{n-(1/2)})$, and this shows that (4.1) is $o((\log(1/\varepsilon))^n)$, which is what we set out to prove.

All that remains, then, is to show that we can always find sets A and B of this form. For this, we will use Lemmas 4 and 5. Lemma 4 gives us a good first initial candidate for A and B. We can let A be equal to the set of (distinct) decreasing u_j's together with elements $u(p_i)$ for each t-free p_i (Recall that all decreasing u_j's have $m_j \geq 1$, by Lemma 5). A possible problem with this is that every increasing u_i, and in particular each possibility for $u(p_i)$, appears at least once in the numerator of (4.11), so that we need to make sure that we really can appropriately choose the $u(p_i)$'s. Nevertheless, as will be shown below, this works for A. B cannot be chosen the same way, however. This is because if u_j is decreasing but u_{j+1} is increasing, then u_j appears exactly once in the numerator of (4.11) and we may have $m_j = 1$, so that u_j cannot be in both A and B. B will have to be formed in a different manner.

To simplify things a bit, let us employ the following notation. Given an increasing u_j, where $u_j - u_{j-1} = p_i$ let $\bar{j}$ be such that $u_{\bar{j}} - u_{\bar{j}+1} = p_i$. For example, in Fig. 4, we would have $u_{\bar{1}} = u_4$, $u_{\bar{2}} = u_2$, and $u_{\bar{4}} = u_5$.

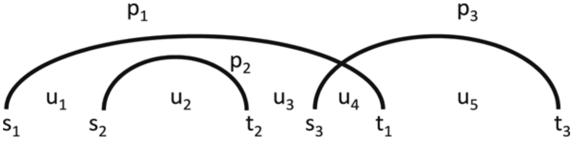

Fig. 4. Sample configuration.

Our goal now is to form a set B which spans the set of all decreasing u_j's, but which contains no decreasing u_j's with $m_j = 1$, for we intend to place those in A. We will begin by creating an increasing collection of sets B_n. Start with the smallest j such that u_j is decreasing. If $m_j = 2$, then let $B_1 = \{u_j\}$. If $m_j = 1$, then we know from Lemma 5 that u_{j+1} is increasing. We then let $B_1 = \{u_{j+1}, u_{\bar{j}}, u_{\bar{j}+1}\}$. We will essentially repeat this for each decreasing u_j. Suppose that the set B_n has already been formed. Let j be as small as possible so that $u_j \notin \text{span}\{B_n\}$ and u_j is decreasing. If $m_j = 2$, then let $B_{n+1} = B_n \bigcup \{u_j\}$. If $m_j = 1$, then let $B_{n+1} = B_n \bigcup \{u_{j+1}, u_{\bar{j}}, u_{\bar{j}+1}\} \setminus \{u_j\}$. The reason for subtracting the element $\{u_j\}$ is that it may already be in the set B_n, having been of the form $u_{\bar{j'}}$ or $u_{\bar{j'}+1}$ for an earlier j'. Repeat this process through all of the decreasing u_j's. The final set obtained, say B_N, will span the set of decreasing u_j's. To see this, suppose to the contrary, and let u_{j_o} be the decreasing u_j with largest j value which is not in the span of B_N. Clearly then $m_{j_o} = 1$, which means that some B_n must have contained u_{j_o+1}, $u_{\bar{j}_o}$, and $u_{\bar{j}_o+1}$. Any of these elements which are increasing must be present in B_N, and any decreasing ones have larger j values than u_{j_o}, which means they are in the span of

B_N. Thus, $u_{j_o} = u_{j_o+1} - (u_{\bar{j}_o} - u_{\bar{j}_o+1})$ is also in the span of B_N, a contradiction. The set B_N also satisfies property (iii) above. This will be shown using the following lemma:

Lemma 6. 1. *If u_i is in B_N then u_i is either decreasing or else neighbors on a decreasing interval (i.e. at least one of u_{i-1} and u_{i+1} is decreasing).*

2. *If u_i, u_{i+1} are both increasing and $u_i \in B_N$ then $u_{i-1} \downarrow$, $m_{i-1} = 1$, and $m_i \geq 1$.*

Proof. If u_i is increasing and in B_N then u_i must be of the form $u_{j+1}, u_{\bar{j}}$, or $u_{\bar{j}+1}$ for some j where u_j is decreasing and $m_j = 1$. It is always true that $u_{\bar{j}+1}$ is decreasing, so this cannot be u_i. (1) is proved by noting that $u_{j+1}, u_{\bar{j}}$ are neighbors to the decreasing intervals $u_j, u_{\bar{j}+1}$ respectively. If, in addition, the situation in (2) arises then u_i cannot be of the form $u_{\bar{j}}$ with $m_j = 1$ since in that case $u_{\bar{j}+1}$ would be decreasing. Thus, u_i is of the form u_{j+1}. In order for u_{j+1} to be included in B' it was necessary that $u_j \downarrow$ and $m_j = 1$. By part 2 in Lemma 5 $m_i \geq 1$. □

The construction of B_N guarantees that if $u_j \in B_N$ and $u_j \downarrow$ then $m_j = 2$. If $u_j \in B_N$ and $u_j \uparrow$ then $m_j \geq 1$ by part 1 of Lemma 5 or part 2 of Lemma 6, depending on whether $u_{j+1} \downarrow$ or $\uparrow$. Thus, B_N satisfies (iii) as claimed. Let $B' = B_N$ and A' be the set of all decreasing u_j's. We know from the discussion above that B' satisfies (i) and (iii). A' clearly satisfies (i), and satisfies (iii) by part 1 of Lemma 5. A' and B' together satisfy (iv) because of the way that B' was constructed, and both span the set of all decreasing u_j's. We need now only extend them to sets A and B which span all of $\{p_1, \ldots, p_n\}$. A' and B' already span the set of all non-t-free p_i's, by the first part of Lemma 4. In the light of the second part of Lemma 4, all that remains is to show that, for any t-free p_i, we can always choose $u_1(p_i), u_2(p_i)$ which contain p_i as a term, and which we may include in A and B respectively without violating rules (iii) and (iv).

Suppose p_i is t-free, and k is chosen as large as possible so that $s_i < s_{i+1} < \cdots < s_{i+k} < t_i$. Let $u_j - u_{j-1} = p_i$. The term $p_i p_{i+1} \cdots p_{i+k}$ in (4.1) becomes $(|u_j| + |u_{j-1}|) \cdots (|u_{j+k}| + |u_{j+k-1}|)$ in (4.11), with $u_j, \ldots, u_{j+k}$ not appearing anywhere else in the numerator. If $k > 1$ we can just note that, upon expanding this expression, the sum of the powers of u_{j+k} and u_{j+k-1} in the numerator is at most two. This means that $m_{j+k} + m_{j+k-1}$ must be at least 2, and we can choose $u_1(p_i), u_2(p_i)$ as some combination of u_{j+k} and u_{j+k-1}. It is possible that u_{j+k} is already in B', and so we must make sure that if $u_1(p_i) = u_{j+k} \neq u_2(p_i)$ that we interchange $u_1(p_i)$ and $u_2(p_i)$, so that u_{j+k} is not in both A and B, which might violate (iv). Note that if $k > 1$ then $u_{j+k-1} \notin B'$ by Lemma 6, since $u_{j+k-1} \uparrow$ and neighbors only on increasing intervals. In the case that $k = 1$ we still have $m_j + m_{j+1} \geq 2$, but now it is possible that both u_j and u_{j+1} are in B', since both neighbor upon intervals which may be decreasing. However, if this is the case then, since $u_j, u_{j+1} \uparrow$, we have by Lemma 6 $u_{j-1} \downarrow$ and $m_{j-1} = 1$. Recall that we have the term $(|u_j| + |u_{j-1}|)(|u_{j+1}| + |u_j|)$ in the numerator, with u_{j-1}, u_j, u_{j+1} appearing nowhere else in the numerator. The sum of the powers of u_{j-1}, u_j, and u_{j+1} is 2, and thus $m_{j-1} + m_j + m_{j+1} = 4$. Since $m_{j-1} = 1$, one of m_j and m_{j+1} is 2. We can then let $u_1(p_i) = u_2(p_i) = u_j$ or u_{j+1}, depending on whether m_j or m_{j+1} is 2. This handles the case $k = 1$. (If $k = 0$ then we would have an isolated interval, and this argument does not work. This is the only place where we have used the fact that we have no isolated intervals.) Doing this for each t-free p_i we create the sets A and B, which are guaranteed by Lemma 4 to satisfy the property (ii). A and B also satisfy properties (i), (ii), and (iv) by construction, so we have completed the proof in the case where no isolated intervals are present.

Now for the isolated intervals case. Recall that the integral which gives us the contribution from this configuration is

$$\int \mathrm{e}^{-\varepsilon \sum p_i^2} \prod (p_i)_1 \prod_{j=1}^{2n-1} \left(\int_{\sum t_j < T} \prod_j \mathrm{e}^{-u_j^2 t_j} \prod_j \mathrm{d}t_j \right) \prod \mathrm{d}p_i. \tag{4.17}$$

As mentioned before, here we cannot replace the integrand with its absolute value, for in that case each isolated interval would contribute a $\sqrt{1/\varepsilon}$ to the integral. Cancelation occurs in the integral, however, since the integrand is positive in some regions and negative in others. It turns out that it is enough to integrate each of the variables corresponding to isolated intervals first, and then to bring the absolute value inside the integral. After we have "removed" the initial set of isolated intervals in this fashion, we will have created a new configuration of intervals, which may again contain isolated intervals. We can remove these isolated intervals by a different method than the one used for the first set. This brings us to a new configuration, which may again have isolated intervals, which we again remove, etc. After a finite number of steps we either have removed all intervals or we have arrived at an arrangement with no isolated intervals. In the second case we are reduced to the case we have already done, and the first is handled easily in a slightly different way.

Let us bring in some definitions in order to make this rigorous. Let our initial configuration of intervals be denoted K_0, and let K_m be the configuration of intervals obtained upon removing the isolated intervals from K_{m-1}. A simple example would be:

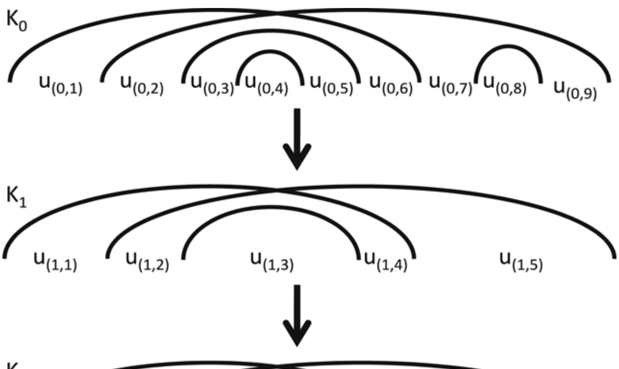

Fig. 5. Removing isolated intervals.

We will say that p_i is in K_m to mean that the interval (s_i, t_i) appears in the configuration K_m, and we will define the *order* of K_m to be the number of p_i's in K_m. Let $u_{(m,1)}, \ldots, u_{(m,n_m)}$ be the u values which appear in the configuration K_m, as the picture (Fig. 5) indicates above. The following notation is necessary in order to write the integral down in the form we desire. Let us define I_m to be the set of all j values corresponding to isolated intervals in K_m; that is, $I_m = \{j : u_{(m,j)} = p_i \text{ where } (s_i, t_i) \text{ is an isolated interval in } K_m\}$. A $\hat{p}$ will refer to the p associated to an isolated interval. That is, if $j \in I_m$ and p_i is the p which appears only in $u_{(m,j)}$, label p_i as $\hat{p}_{m,j}$. As an example, in the following configuration (Fig. 6) we would have $I_0 = \{2, 5\}$, $\hat{p}_{(0,2)} = p_2$, and $\hat{p}_{(0,5)} = p_4$.

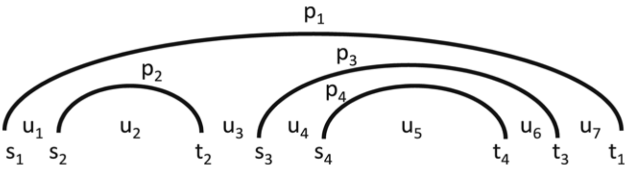

Fig. 6. Sample configuration.

We can bound (4.17) by

$$\int \prod \mathrm{e}^{-\varepsilon p_i^2}|p_i| \times \left(\int \prod_{j\notin I_0} \mathrm{e}^{-u_{(0,j)}^2 t_j} \left| \prod_{j\in I_0} \int \int \mathrm{e}^{-\varepsilon \hat{p}_{(0,j)}^2} (\hat{p}_{(0,j)})_1 \mathrm{e}^{-u_{(0,j)}^2 t_j} \mathrm{d}t_j \mathrm{d}\hat{p}_{(0,j)} \right| \prod_{j\notin I_0} \mathrm{d}t_j \right) \prod \mathrm{d}p_i. \tag{4.18}$$

The first and last products are over all i such that $p_i \neq \hat{p}_{(0,j)}$ for all j. We will get a good bound on the $\mathrm{d}t_j \mathrm{d}\hat{p}_j$ integrals. Note that we have suppressed the region of integration in t_j, since it may be quite complicated. We do know that the upper limit of integration is bounded above by T, and this allows us to get a sufficient bound, as the following lemma shows.

Lemma 7. *For any a with $0 < a < T$ and any $k \in \mathbf{R}^2$, we have*

$$\left| \int \int_0^a \mathrm{e}^{-\varepsilon p^2} p_1 \mathrm{e}^{-(p+k)^2 t} \mathrm{d}t \mathrm{d}p \right| = |k| O(\log(1/\varepsilon)) \tag{4.19}$$

independently of a.

Proof.

$$\int \int_0^a \mathrm{e}^{-\varepsilon p^2} p_1 \mathrm{e}^{-(p+k)^2 t} \mathrm{d}t \mathrm{d}p = \int_0^a \left(\int \mathrm{e}^{-\varepsilon p_1^2} p_1 \mathrm{e}^{-(p_1+k_1)^2 t} \mathrm{d}p_1 \int \mathrm{e}^{-\varepsilon p_2^2} \mathrm{e}^{-(p_2+k_2)^2 t} \mathrm{d}p_2 \right) \mathrm{d}t. \tag{4.20}$$

Now, for $p, k \in R^2$, $(p+k)^2 = p^2 + k^2 + 2p \cdot k$, so this is

$$\begin{aligned} &\int_0^a \mathrm{e}^{-k^2 t} \left(\int \mathrm{e}^{-\varepsilon p_1^2} p_1 \mathrm{e}^{-(p_1^2+2p_1k_1)t} \mathrm{d}p_1 \int \mathrm{e}^{-\varepsilon p_1^2} \mathrm{e}^{(-p_2^2+2p_2k_2)t} \mathrm{d}p_2 \right) \mathrm{d}t \\ &= \int_0^a \mathrm{e}^{-k^2 t} \mathrm{e}^{\frac{k^2 t^2}{\varepsilon+t}} \left(\int \mathrm{e}^{-(\varepsilon+t)(p_1+\frac{k_1 t}{\varepsilon+t})^2} p_1 \mathrm{d}p_1 \int \mathrm{e}^{-(\varepsilon+t)(p_2+\frac{k_2 t}{\varepsilon+t})^2} \mathrm{d}p_2 \right) \mathrm{d}t \\ &= \int_0^a \mathrm{e}^{-k^2 t} \mathrm{e}^{\frac{k^2 t^2}{\varepsilon+t}} \left(\int \mathrm{e}^{-(\varepsilon+t)p_1^2} (p_1 - \frac{k_1 t}{\varepsilon+t}) \mathrm{d}p_1 \int \mathrm{e}^{-(\varepsilon+t)p_2^2} \mathrm{d}p_2 \right) \mathrm{d}t. \end{aligned} \tag{4.21}$$

We now split the p_1 integral into two pieces, and we see that the first one,

$$\int \mathrm{e}^{-(\varepsilon+t)p_1^2} p_1 \mathrm{d}p_1, \tag{4.22}$$

is 0 by symmetry (this is what will give us the extra convergence). We use the fact that, for $d = 1, 2$ we have

$$\int e^{-(\varepsilon+t)p_{\mathrm{d}}^2} \mathrm{d}p_d = \frac{c}{\sqrt{\varepsilon + t}} \tag{4.23}$$

for some constant c. We will also replace $\frac{t}{\varepsilon+t}$ and $e^{-k^2 t} e^{\frac{k^2 t^2}{\varepsilon+t}}$ by the trivial bound of 1. This shows us that we can bound (4.21) by

$$c^2 |k_1| \int_0^a \frac{1}{\varepsilon + t} \mathrm{d}t. \tag{4.24}$$

Since $a < T$, this is $|k| O(\log(1/\varepsilon))$, independently of a. $\square$

We integrate the $\hat{p}_{0,j}$'s first, and by the previous lemma each one gives $|u_{0,j-1}| O(\log(1/\varepsilon))$ ($|u_{0,j-1}|$ is the u_j which appears immediately before and after the isolated interval corresponding to $\hat{p}_{0,j}$). (4.18) is thus

$$O(\log(1/\varepsilon))^{|I_0|} \int \prod_{p_i \neq \hat{p}_{(0,j)} \forall j} e^{-\varepsilon p_i^2} |p_i| \prod_{(j+1) \in I_0} |u_{(0,j)}|$$
$$\times \left(\int \prod_{j \notin I_0} e^{-u_{(0,j)}^2 t_j} \prod_{j \notin I_0} \mathrm{d}t_j \right) \prod_{p_i \neq \hat{p}_{(0,j)} \forall j} \mathrm{d}p_i. \tag{4.25}$$

Since the integrand is now positive we can extend the region of integration for the t_i's to be $0 < t_i < T$ and use (4.2) to bound (4.25) by

$$O(\log(1/\varepsilon))^{|I_0|} \int \prod_{p_i \neq \hat{p}_{(0,j)} \forall j} e^{-\varepsilon p_i^2} |p_i| \prod_{(j+1) \in I_0} |u_{(0,j)}| \prod_{j \notin I_0} \frac{1}{1 + u_{(0,j)}^2} \prod_{p_i \neq \hat{p}_{(0,j)} \forall j} \mathrm{d}p_i. \tag{4.26}$$

Suppose that $u_{m,j}$ is an isolated interval in K_m. Then $u_{m,j-1} = u_{m,j+1}$. We will say in this case that $u_{m,j-1}$ contains $u_{m,j}$. If $u_{m,j} = u_{m',j'}$, where $m > m'$, and $u_{m',j'+1}$ is isolated in $K_{m'}$, we will also say that $u_{m,j}$ contains $u_{m',j'+1}$. We will let $l_{m,j}$ denote the total number of isolated intervals which the interval $u_{m,j}$ contained in all $K_{m'}$'s, where $m' < m$. Each $u_{1,j}$ which contained one or more isolated intervals in K_0 will appear to a power $l_{1,j}$ in the numerator of (4.26) as a result of Lemma 7, but the term $(1 + u_{1,j}^2)$ will also appear an extra $l_{1,j}$ times in the denominator. We see that (4.26) is

$$O(\log(1/\varepsilon))^{|I_0|} \int e^{-\varepsilon \sum_{p_i \in K_1} p_i^2} \prod_{p_i \in K_1} |p_i| \prod_{1 \le j \le n_1} |u_{(1,j)}|^{l_{1,j}} \frac{1}{(1 + u_{(1,j)}^{2+2l_{1,j}})} \prod_{p_i \in K_1} \mathrm{d}p_i. \tag{4.27}$$

We must have some idea how the integral (4.26) can be bounded as we remove successive stages of isolated intervals, and Lemma 9 below gives us that. The following lemma prepares us to prove Lemma 9.

Lemma 8.

$$\int e^{-\varepsilon p^2} |p| \frac{1}{(1 + |k + p|)^m} \mathrm{d}p \tag{4.28}$$

is $(1 + |k|) O(1) + O(\log(1/\varepsilon))$ *if* $m = 3$, *and is* $(1 + |k|) O(1)$ *if* $m > 3$.

Proof. (4.28) is bounded by

$$\int \mathrm{e}^{-\varepsilon(p-k)^2}(|p|+|k|)\frac{1}{(1+|p|)^m}\mathrm{d}p. \tag{4.29}$$

Divide this into two integrals. The one with $|k|$ in the numerator is bounded by

$$|k|\int \frac{1}{(1+|p|)^m}\mathrm{d}p = |k|O(1). \tag{4.30}$$

The other is bounded by

$$c\int \mathrm{e}^{-\varepsilon(p-k)^2}\frac{1}{1+|p|^{m-1}}\mathrm{d}p. \tag{4.31}$$

Again if $m > 3$ this is $O(1)$. If $m = 3$, divide the region into $\{|p| > 2|k|\}$ and $\{|p| < 2|k|\}$. On $\{|p| > 2|k|\}$ we can bound the integral by

$$\int \mathrm{e}^{\varepsilon p^2/2}\frac{1}{1+p^2} = O(\log(1/\varepsilon)) \tag{4.32}$$

by Lemma 2. On $\{|p| < 2|k|\}$ we can bound it by

$$\int_{|p|<2|k|}\frac{1}{1+p^2}\mathrm{d}p \le \log(|k|+1) \le |k|. \tag{4.33}$$

These bounds combine to prove the lemma. $\square$

Lemma 9. *Suppose that K_m contains isolated intervals. Then (4.17) is*

$$O(\log(1/\varepsilon))^{|I_0|+\cdots+|I_m|}\int \mathrm{e}^{-\varepsilon\sum_{p_i\in K_{m+1}} p_i^2}\prod_{p_i\in K_{m+1}}|p_i|$$
$$\times\prod_{1\le j\le n_{m+1}}(1+|u_{(m+1,j)}|)^{l_{m+1,j}}\frac{1}{(1+u_{(m+1,j)}^{2+2l_{m+1,j}})}\prod_{p_i\in K_{m+1}}\mathrm{d}p_i. \tag{4.34}$$

Proof. By induction. We know that it is true for $m = 0$ (see (4.27)). Assume that it is true for $m-1$, so (4.17) is

$$O(\log(1/\varepsilon))^{|I_0|+\cdots+|I_{m-1}|}\int \mathrm{e}^{-\varepsilon\sum_{p_i\in K_m} p_i^2}\prod_{p_i\in K_m}|p_i|$$
$$\times\prod_{1\le j\le n_m}(1+|u_{(m,j)}|)^{l_{m,j}}\frac{1}{(1+u_{(m,j)}^{2+2l_{m,j}})}\prod_{p_i\in K_m}\mathrm{d}p_i. \tag{4.35}$$

We will integrate the variables in K_m corresponding to isolated intervals. We can rewrite the integral in (4.35) as

$$\int \mathrm{e}^{-\varepsilon\sum_{p_i\in K_m,i\notin\hat{I}_m} p_i^2}\prod_{p_i\in K_m,i\notin\hat{I}_m}|p_i|\prod_{j\notin I_m}\frac{(1+|u_{(m,j)}|)^{l_{m,j}}}{(1+u_{(m,j)}^{2+2l_{m,j}})}$$
$$\times\left(\prod_{j\in I_m}\int|\hat{p}_{m,j}|\mathrm{e}^{-\varepsilon\hat{p}_{m,j}}\frac{(1+|u_{(m,j)}|)^{l_{m,j}}}{(1+u_{(m,j)}^{2+2l_{m,j}})}\mathrm{d}\hat{p}_{m,j}\right)\prod_{p_i\in K_m i\notin\hat{I}_m}\mathrm{d}p_i. \tag{4.36}$$

It is simple to verify that

$$\frac{(1+|u_{(m,j)}|)^{l_{m,j}}}{(1+u_{(m,j)}^{2+2l_{m,j}})} \leq K\frac{1}{(1+u_{(m,j)}^{2+l_{m,j}})}, \tag{4.37}$$

for some constant K depending on $l_{m+1,j}$. Each $\hat{p}_{m,j}$ integral is

$$(1+|u_{m,j}-\hat{p}_{m,j}|)O(\log(1/\varepsilon)) \tag{4.38}$$

by Lemma 8. Plugging this into (4.36) and relabeling the u's with index $m+1$ instead of m gives (4.34). □

To complete the proof of Proposition 2, let us consider several cases. Recall that *order* refers to how many intervals $[s_i, t_i]$ make up a configuration.

Case 1: There is a K_m of order greater than or equal to 3 which contains no isolated intervals.

In this case our integral in (4.34) is almost the same as what would have been obtained if we had started with the configuration K_m. The only difference is the presence of the extra powers $l_{m,j}$, which in fact cause greater convergence. Thus, by what we did earlier in this section, the remaining integral is $o(\log(1/\varepsilon))^{|K_m|}$. Since

$$|I_0|+\cdots+|I_{m-1}|+|K_m| = n, \tag{4.39}$$

we see that (4.17) is $o(\log(1/\varepsilon))^n$, which is what we set out to prove.

Case 2: There is a K_m of order 2 with no isolated intervals.

As before we get $O(\log(1/\varepsilon))^{|I_0|+\cdots+|I_{m-1}|}$ times an integral nearly identical to what we would have had if starting with K_m. Again there will be extra factors which aid convergence. The integral in question can be bounded by one of the following integrals:

$$\begin{gathered}\int\int \frac{1}{(1+|p|)^2(1+|q|)^2(1+|p+q|)^3}\mathrm{e}^{-\varepsilon(p^2+q^2)}|p||q|\mathrm{d}p\mathrm{d}q \\ \int\int \frac{1}{(1+|p|)^2(1+|q|)^3(1+|p+q|)^2}\mathrm{e}^{-\varepsilon(p^2+q^2)}|p||q|\mathrm{d}p\mathrm{d}q \\ \int\int \frac{1}{(1+|p|)^3(1+|q|)^2(1+|p+q|)^2}\mathrm{e}^{-\varepsilon(p^2+q^2)}|p||q|\mathrm{d}p\mathrm{d}q.\end{gathered} \tag{4.40}$$

Therefore the following lemma completes the proof in this case.

Lemma 10. *Each of the integrals in (4.40) is* $o(\log(1/\varepsilon))^2$

Proof. This is fairly straightforward to calculate using Lemmas 2 and 8. For example, by the Cauchy–Schwarz inequality and symmetry we can bound the first integral by

$$k\int\int \frac{1}{(1+|p|)^4(1+|p+q|)^3}\mathrm{e}^{-\varepsilon(p^2+q^2)}|p||q|\mathrm{d}p\mathrm{d}q. \tag{4.41}$$

The dq integral is $(1+|p|)O(1)+O(\log(1/\varepsilon))$ by Lemma 8, and thus (4.41) is

$$kO(1)\int\int \frac{1}{(1+|p|)^2}\mathrm{e}^{-\varepsilon p^2}\mathrm{d}p + O(\log(1/\varepsilon))\int\int \frac{1}{(1+|p|)^3}\mathrm{e}^{-\varepsilon p^2}\mathrm{d}p, \tag{4.42}$$

which is $O(\log(1/\varepsilon))$ by Lemma 2 and the fact that $\frac{1}{(1+|p|)^3} \in L^1$.

The second and third integrals are identical with p and q interchanged, so we need only do one, let us say the second one. This is bounded by

$$k \int\int \frac{1}{(1+|p|)(1+|q|)^2(1+|p+q|)^2} \mathrm{e}^{-\varepsilon(p^2+q^2)} \mathrm{d}p\mathrm{d}q. \tag{4.43}$$

By Cauchy–Schwarz, this is bounded by

$$\left(\int\int \frac{1}{(1+|p|)^2(1+|q|)^2} \mathrm{e}^{-\varepsilon(p^2+q^2)} \mathrm{d}p\mathrm{d}q\right)^{1/2} \times \left(\int\int \frac{1}{(1+|q|)^2(1+|p+q|)^4} \mathrm{e}^{-\varepsilon(p^2+q^2)} \mathrm{d}p\mathrm{d}q\right)^{1/2}. \tag{4.44}$$

This first integral is $O(\log(1/\varepsilon))^2$ by Lemma 2, and the second one is $O(\log(1/\varepsilon))$, using Lemma 2 in conjunction with the fact that

$$\int \frac{1}{(1+|p+q|)^4} \mathrm{d}p = O(1). \tag{4.45}$$

As a simple alternate proof, one can recall our proof for the case with no isolated intervals where we constructed the sets A and B. Here it is simple to verify in each case that we can form two sets with the same properties. The lemma is then proved by the reasoning in steps (4.13) through (4.16). □

Case 3: There is a K_m consisting of just one interval.

Here we must examine in closer detail the proof of Lemma 9. First of all, if there was ever an isolated interval in some $K_{m'}$ which contained two or more isolated intervals in K_m's with $m < m'$, then the variable corresponding to that interval, say $u_{(m',j)}$, would have had $l_{m',j} \geq 2$. In that case, by Lemma 8, the contribution to (4.36) of the $\hat{p}_{m',j}$ integral is $O(1)(1+|u_{(m',j)} - \hat{p}_{m',j})|$. We see that we can replace the term $O(\log(1/\varepsilon))^{|I_0|+\cdots+|I_{m-1}|}$ in (4.34) with $o(\log(1/\varepsilon))^{|I_0|+\cdots+|I_{m-1}|}$, which will finish the proof. Thus we need only consider the case where $s_1 < s_2 < \cdots < s_n < t_n < \cdots < t_2 < t_1$. In this case, consider what happens as we remove the first three intervals (recall that we are assuming that there are at least three intervals). After removing (s_n, t_n) and then (s_{n-1}, t_{n-1}) we have

$$O(\log(1/\varepsilon)) \int \mathrm{e}^{-\varepsilon \sum_{1\le i\le n-2} p_i^2} \prod_{1\le i\le n-2} |p_i| \times (1+|u_{(2,n-2)}| + O(\log(1/\varepsilon))) \prod_{1\le j\le n_2} \frac{1}{(1+u^2_{(2,j)})} \prod_{1\le i\le n-2} \mathrm{d}p_i. \tag{4.46}$$

Note that $u_{2,n-2} = p_1 + \cdots + p_{n-2}$. We can expand this into two integrals, namely

$$O(\log(1/\varepsilon)) \int \mathrm{e}^{-\varepsilon \sum_{1\le i\le n-2} p_i^2} \prod_{1\le i\le n-2} |p_i|(1+|u_{(2,n-2)}|) \prod_{1\le j\le n_2} \frac{1}{(1+u^2_{(2,j)})} \prod_{1\le i\le n-2} \mathrm{d}p_i \tag{4.47}$$

and

$$O(\log(1/\varepsilon))^2 \int e^{-\varepsilon \sum_{1\le i\le n-2} p_i^2} \prod_{1\le i\le n-2} |p_i| \prod_{1\le j\le n_2} \frac{1}{(1+u_{(2,j)}^2)} \prod_{1\le i\le n-2} \mathrm{d}p_i. \tag{4.48}$$

The integral in (4.47) is $O(\log(1/\varepsilon))^{n-2}$ by the same technique as was used to prove Lemma 9. Thus, (4.47) is $O(\log(1/\varepsilon))^{n-1}$. As for (4.48), when we remove the next interval, (s_{n-2}, t_{n-2}), we have no powers of $|u_{2,n-2}|$ in the numerator, and by Lemma 8 we do not pick up an $O(\log(1/\varepsilon))$ term. Thus, (4.48) is $o(\log(1/\varepsilon))^n$ as well. This completes the proof of Proposition 2.

Remark. Note that in every case in the previous two sections we were able to obtain convergence as $\varepsilon \longrightarrow 0$ which was independent of T, provided that T was bounded by some large $M > 0$. Thus, if we restrict to $T < M$, we have uniform convergence of the moments.

5. Completing the proof

All that remains is to prove that the processes $\alpha'_\varepsilon(S)$ are tight and that the limit process has independent increments. Both are essentially corollaries of the following lemma:

Lemma 11. *If $b \le c$, then $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([a,b]\times[c,d]) \longrightarrow 0$ in L^n as $\varepsilon \longrightarrow 0$, for any $n \ge 3$. Furthermore, for $a, b, c, d < T$, this convergence is uniform.*

Proof. To compute $E[\alpha'_\varepsilon([a,b]\times[c,d])]^n$, we multiply the integrals together as before (see (2.4)). Now, however, we have $s_i \le b \le c \le t_i$ for all i, and it follows from this that the only configurations of intervals that can appear here are ones containing just one component of order n. We have shown that these components contribute $o(\log(1/\varepsilon))^n$ to the nth moment, and this is enough to prove the first part of the lemma. The remark at the end of Section 4 demonstrates that the convergence is uniform, which is the second part of the lemma. $\square$

Now that we have this lemma, we can show that the processes $\alpha'_\varepsilon(S)(\log(1/\varepsilon))^{-1}$ are tight. We will show that

$$E[(\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon(R) - \alpha'_\varepsilon(S))]^{2n} \le k(R-S)^n, \tag{5.1}$$

where k depends on $n \ge 2$ but can be chosen independently of ε and $R, S < T$. This will prove tightness by, for example, Theorem 12.3 in [1]. We can rewrite the left-hand side of (5.1) as

$$E\left[(\log(1/\varepsilon))^{-1}\left(\alpha'_\varepsilon([0,R]\times[R,S]) + \alpha'_\varepsilon\left(D_T \bigcap \{s,t \ge R\}\right)\right)\right]^{2n}. \tag{5.2}$$

We know by the lemma that $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([0,R]\times[R,S]) \longrightarrow 0$ uniformly in L^{2n}, so we need only show that

$$kE\left[(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon\left(D_T \bigcap \{s,t \ge R\}\right)\right]^{2n} \le k(R-S)^n \tag{5.3}$$

independently of ε and $R, S < T$. Suppressing $(\log(1/\varepsilon))^{-1}$ for the time being, this is given by

$$\frac{(-1)^n}{(2\pi)^{4n}} \int\int_{D_T^{2n}\bigcap\{s,t\ge S\}} e^{-\varepsilon\sum_j p_j^2} \prod_{j=1}^n p_{j,1} E\left[\prod_{j=1}^n e^{\mathrm{i}p_j(X_{t_j}-X_{s_j})}\right] \prod_{j=1}^n \mathrm{d}s_j \mathrm{d}t_j \mathrm{d}^2 p_j. \tag{5.4}$$

If we rewrite $X_{t_j} - X_{s_j}$ as $(X_{t_j} - X_S) - (X_{s_j} - X_S)$, and let $\beta_t = X_{S+t} - X_S$ be a new Brownian motion this is

$$\frac{(-1)^n}{(2\pi)^{4n}} \int\int_{D_{(T-S)^{2n}}} \mathrm{e}^{-\varepsilon \sum_j p_j^2} \prod_{j=1}^{n} p_{j,1} E\left[\prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}p_j(\beta_{t_j} - \beta_{s_j})}\right] \prod_{j=1}^{n} \mathrm{d}s_j \mathrm{d}t_j \mathrm{d}^2 p_j, \tag{5.5}$$

which is equal to(reinserting $(\log(1/\varepsilon))^{-1}$)

$$kE[(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T-S)]^{2n}. \tag{5.6}$$

This is $O(1)|T-S|^n$, as we showed earlier. This establishes tightness. We can write

$$\begin{aligned}&(\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon(T) - \alpha'_\varepsilon(S))\\ &\quad = (\log(1/\varepsilon))^{-1}\left(\alpha'_\varepsilon([0,S]\times[S,T]) + \alpha'_\varepsilon\left(D_T \bigcap \{s,t \geq S\}\right)\right).\end{aligned} \tag{5.7}$$

Since $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([0,S]\times[S,T]) \longrightarrow 0$ and $\alpha'_\varepsilon(D_T \bigcap\{s,t \geq S\}) \in \bigcup\{\sigma(X_t - X_s) : S \leq s,t \leq T\}$, we see that $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T)$ has asymptotically independent increments. This shows that the limit process, W_T, has independent increments, and completes the proof of Theorem 1.

6. Symmetric stable processes

We will now prove Theorem 2. The proof of this theorem is, naturally, very similar to the proof in the Brownian motion case, so we will in many cases just refer to steps undertaken in the previous proof. In particular, the general outline (Sections 2 and 5) is identical in both cases; the only difference lies in some of the calculations.

The main difficulty is in showing that the integrals corresponding to components of order two converge. X_t is a symmetric stable process of index β where $1 < \beta < 2$. The density of X_t is given by

$$f_\varepsilon(x) = \frac{1}{(2\pi)^2}\int \mathrm{e}^{\mathrm{i}px - \varepsilon p^\beta} \mathrm{d}^2 p. \tag{6.1}$$

Thus,

$$f'_\varepsilon(x) = \frac{\mathrm{i}}{(2\pi)^2}\int p_1 \mathrm{e}^{\mathrm{i}px - \varepsilon p^\beta} \mathrm{d}^2 p. \tag{6.2}$$

Proceeding as in Section 3, the first integral is

$$\begin{aligned}&\int\int \frac{(1-\mathrm{e}^{-p^\beta})}{p^\beta}\frac{(1-\mathrm{e}^{-q^\beta})}{q^\beta}\frac{(1-\mathrm{e}^{-(p+q)^\beta})}{(p+q)^\beta}\mathrm{e}^{-\varepsilon(p^\beta+q^\beta)} p_1 q_1 \mathrm{d}p\mathrm{d}q\\ &\quad = \varepsilon^{3-6/\beta}\int\int \frac{(1-\mathrm{e}^{-p^\beta/\varepsilon})}{p^\beta}\frac{(1-\mathrm{e}^{-q^\beta/\varepsilon})}{q^\beta}\frac{(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon})}{(p+q)^\beta}\mathrm{e}^{-(p^\beta+q^\beta)} p_1 q_1 \mathrm{d}p\mathrm{d}q.\end{aligned} \tag{6.3}$$

In order to prove that this integral converges as $\varepsilon \longrightarrow 0$, it is enough to show that

$$\int\int \frac{1}{p^{\beta-1}}\frac{1}{q^{\beta-1}}\frac{1}{(p+q)^\beta}\mathrm{e}^{-(p^\beta+q^\beta)}\mathrm{d}p\mathrm{d}q \tag{6.4}$$

converges, and then to apply the dominated convergence theorem. We need only consider the integral over $\{|p|, |q| < 1\}$, for in order to evaluate the integral over, say, $A = \{|p| > 1\}$ we may divide A into the disjoint union of $B = \{|q| < 1/2\} \bigcap A$, $C = \{|p+q| < 1/2\} \bigcap A$, and $D = A - (B \bigcup C)$. The integrals over B and C are both bounded because $\beta < 2$ i.e. we have only integrable singularities. And the integral over D is bounded by a constant times

$$\int\int \mathrm{e}^{-(p^\beta + q^\beta)} \mathrm{d}p\mathrm{d}q < \infty. \tag{6.5}$$

Therefore, we must consider the integral

$$\int\int_{\{|p|,|q|<1\}} \frac{1}{p^{\beta-1}} \frac{1}{q^{\beta-1}} \frac{1}{(p+q)^\beta} \mathrm{d}p\mathrm{d}q. \tag{6.6}$$

We manipulate the integral as follows:

$$\begin{aligned} &\int_{|p|<1} \frac{1}{p^{\beta-1}} \int_{\{|q|<1\}} \frac{1}{q^{\beta-1}} \frac{1}{(p+q)^\beta} \mathrm{d}q\mathrm{d}p \\ &\quad = \int_{|p|<1} \frac{1}{p^{3\beta-2}} \int_{\{|q|<1\}} \frac{1}{(q/|p|)^{\beta-1}} \frac{1}{(p/|p| + q/|p|)^\beta} \mathrm{d}q\mathrm{d}p. \end{aligned} \tag{6.7}$$

The argument of p (thought of as a complex number) is irrelevant, so we may replace $p/|p|$ by 1, and substitute $q' = q/|p|$ to get

$$\int_{|p|<1} \frac{1}{p^{3\beta-4}} \left(\int_{\{|q|<1/|p|\}} \frac{1}{q^{\beta-1}} \frac{1}{(1+q)^\beta} \mathrm{d}q \right) \mathrm{d}p. \tag{6.8}$$

If $\beta > 3/2$ then the dq integral is bounded independently of $|p|$ (since then $\frac{1}{q^{\beta-1}} \frac{1}{(1+q)^\beta} \in L^1$), so that (6.6) is bounded by a constant times

$$\int_{\{|p|<1\}} \frac{1}{p^{3\beta-4}} \mathrm{d}p, \tag{6.9}$$

which is finite, as $\beta < 2$. If $\beta < 3/2$ (resp. $\beta = 3/2$), then the dq integral in (6.8) is $O(|p|)^{2\beta-3}$ (resp. $O(|\log|p||)$), so that (6.6) is bounded by a constant times

$$\int_{\{|p|<1\}} \frac{1}{p^{\beta-1}} \mathrm{d}p \tag{6.10}$$

when $\beta < 3/2$ and

$$\int_{\{|p|<1\}} \frac{|\log(|p|)|}{p^{1/2}} \mathrm{d}p \tag{6.11}$$

when $\beta = 3/2$. These integrals are both finite.

The second configuration of intervals gives rise to the following:

$$\int\int \frac{(1 - \mathrm{e}^{-p^\beta/\varepsilon})^2}{p^{2\beta}} \frac{(1 - \mathrm{e}^{-(p+q)^\beta/\varepsilon})}{(p+q)^\beta} \mathrm{e}^{-(p^\beta + q^\beta)} p_1 q_1 \mathrm{d}p\mathrm{d}q. \tag{6.12}$$

This integral is more difficult as for some β the integrand is not in L^1 were we to remove the terms involving ε (there is a non-integrable singularity at $p = 0$ when $\beta \geq 3/2$). We will first show that (6.12) is bounded independently of ε. We isolate the dq integral:

$$\int \frac{(1 - \mathrm{e}^{-(p+q)^\beta/\varepsilon})}{(p+q)^\beta} \mathrm{e}^{-q^\beta} q_1 \mathrm{d}q. \tag{6.13}$$

We will show that this is $|p|O(1)$ (the O here refers to ε). Because we will refer to this result later, we isolate it as a lemma (which we state in slightly greater generality).

Lemma 12. *For any a with $0 < a < T$ and any $p \in \mathbf{R}^2$, we have*

$$\left| \int \frac{1 - \mathrm{e}^{-(p+q)^\beta a/\varepsilon}}{(p+q)^\beta} \mathrm{e}^{q^\beta} q_1 \mathrm{d}q \right| = |p|O(1), \tag{6.14}$$

independently of a.

Proof. We can drop the $\mathrm{e}^{-(p+q)^\beta a/\varepsilon}$ term. (6.14) is bounded by

$$\left| \int \frac{1}{q^\beta} \mathrm{e}^{-(q-p)^\beta} (q_1 - p_1) \mathrm{d}q \right|. \tag{6.15}$$

Expand the $(q_1 - p_1)$ term. The second term is bounded by

$$|p| \int \frac{1}{q^\beta} \mathrm{e}^{-(q-p)^\beta} \mathrm{d}q. \tag{6.16}$$

The integrand is bounded by the function

$$\frac{1}{q^\beta} 1_{\{|q|<1\}} + \mathrm{e}^{-(q-p)^\beta} 1_{\{|q|\geq 1\}}, \tag{6.17}$$

which is bounded in L^1 independently of p. Thus, (6.16) is $|p|O(1)$. To bound the first term we subtract

$$\int \frac{1}{q^\beta} \mathrm{e}^{-q^\beta} q_1 \mathrm{d}q, \tag{6.18}$$

which is 0 by symmetry. This gives us

$$\left| \int \frac{1}{q^\beta} (\mathrm{e}^{-(q-p)^\beta} - \mathrm{e}^{-q^\beta}) q_1 \mathrm{d}q \right| \leq \int \frac{1}{q^\beta} |\mathrm{e}^{-(q-p)^\beta} - \mathrm{e}^{-q^\beta}||q| \mathrm{d}q. \tag{6.19}$$

We split this up into the integral over the region $\{|q| < 2|p|\}$ and $\{|q| \geq 2|p|\}$. The integral over the first region is bounded by

$$k \int_{\{|q|<2|p|\}} |q|^{1-\beta} \mathrm{d}q = k \int_0^{2|p|} r^{2-\beta} \mathrm{d}r = k|p|^{3-\beta}.$$

Here k is a constant which may change from line-to-line. This is $(|p| + |p|^2)O(1)$. On the region $\{|q| \geq 2|p|\}$ suppose first that $\mathrm{e}^{-(q-p)^\beta} \geq \mathrm{e}^{-q^\beta}$. Then

$$\begin{aligned} |\mathrm{e}^{-(q-p)^\beta} - \mathrm{e}^{-q^\beta}| &\leq \mathrm{e}^{-(|q|-|p|)^\beta} - \mathrm{e}^{-q^\beta} \\ &= \beta \int_{|q|-|p|}^{|q|} x^{\beta-1} \mathrm{e}^{-x^\beta} \mathrm{d}x \leq k|p||q|^{\beta-1} \mathrm{e}^{-(|q|-|p|)^\beta}. \end{aligned} \tag{6.20}$$

The last inequality is the length of the interval being integrated over multiplied by a term which bounds the integrand. Plugging this into (6.19) gives a bound of

$$k|p|\int_{|q|\geq 2|p|} \mathrm{e}^{-(|q|-|p|)^\beta}\mathrm{d}q \leq k|p|\int_{|q|\geq 2|p|} \mathrm{e}^{-(|q|/2)^\beta}\mathrm{d}q = O(|p|). \tag{6.21}$$

In the case $\mathrm{e}^{-(q-p)^\beta} < \mathrm{e}^{-q^\beta}$ we have

$$\begin{aligned}|\mathrm{e}^{-(q-p)^\beta} - \mathrm{e}^{-q^\beta}| &\leq \mathrm{e}^{-|q|^\beta} - \mathrm{e}^{-(|q|+|p|)^\beta} \\ &= \beta\int_{|q|}^{|q|+|p|} x^{\beta-1}\mathrm{e}^{-x^\beta}\mathrm{d}x \leq k|p|(|q|+|p|)^{\beta-1}\mathrm{e}^{-|q|^\beta}.\end{aligned} \tag{6.22}$$

Since $|q| \geq 2|p|$ this is $k|p||q|^{\beta-1}\mathrm{e}^{-|q|^\beta}$ Thus, the contribution to (6.19) of this region is bounded by

$$k|p|\int \mathrm{e}^{-|q|^\beta}\mathrm{d}q = |p|O(1). \tag{6.23}$$

This shows that

$$\int \frac{1}{q^\beta}\mathrm{e}^{-(q-p)^\beta} q_1 \mathrm{d}q = (|p| + |p|^2)O(1). \tag{6.24}$$

It is also $O(1)$, however, since the integrand is bounded by

$$\frac{1}{q^{\beta-1}}1_{\{|q|<1\}} + \mathrm{e}^{-(q-p)^\beta}, \tag{6.25}$$

which is bounded in L^1 independently of p. So (6.24) is $|p|O(1)$ for p small, and $O(1)$ for p large. We conclude that (6.24) is $|p|O(1)$ for all p. □

This lemma allows us to see that (6.12) is bounded by

$$k\int \frac{(1-\mathrm{e}^{-p^\beta/\varepsilon})^2}{p^{2\beta}}\mathrm{e}^{-p^\beta}|p|^2\mathrm{d}p. \tag{6.26}$$

The extra powers of p in the numerator are enough to convert our singularity at 0 into an integrable one, and it follows that (6.26) is bounded by

$$k\int \frac{1}{p^{2\beta}}\mathrm{e}^{-p^\beta}|p|^2\mathrm{d}p < \infty. \tag{6.27}$$

We have showed that (6.12) is bounded independently of ε. This alone does not show that (6.12) converges. However, convergence is proved using the same ideas, as follows. Let the value of (6.12) be denoted by $A(\varepsilon)$. We will show that, for any $\delta > 0$, there is an $\varepsilon' > 0$ such that if $0 < \varepsilon_1, \varepsilon_2 < \varepsilon'$ then $|A(\varepsilon_1) - A(\varepsilon_2)| < \delta$. This will prove convergence. We will assume below that $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon'$. We have

$$\begin{aligned}&A(\varepsilon_1) - A(\varepsilon_2) \\ &\quad = \int\int\left(\frac{(1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2}{p^{2\beta}}\frac{(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_1})}{(p+q)^\beta} - \frac{(1-\mathrm{e}^{-p^\beta/\varepsilon_2})^2}{p^{2\beta}}\frac{(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_2})}{(p+q)^\beta}\right) \\ &\qquad \times \mathrm{e}^{-(p^\beta+q^\beta)}p_1q_1\mathrm{d}p\mathrm{d}q.\end{aligned} \tag{6.28}$$

We will rewrite the difference

$$(1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_1})-(1-\mathrm{e}^{-p^\beta/\varepsilon_2})^2(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_2}) \tag{6.29}$$

as

$$\begin{aligned}&(1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2[(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_1})-(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_2})]\\ &\quad+[(1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2-(1-\mathrm{e}^{-p^\beta/\varepsilon_2})^2](1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_2}),\end{aligned} \tag{6.30}$$

and handle each term in this sum separately. The first one gives rise to the integral

$$\int\int \frac{(1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2}{p^{2\beta}}\frac{(\mathrm{e}^{-(p+q)^\beta/\varepsilon_2}-\mathrm{e}^{-(p+q)^\beta/\varepsilon_1})}{(p+q)^\beta}\mathrm{e}^{-(p^\beta+q^\beta)}p_1q_1\mathrm{d}p\mathrm{d}q. \tag{6.31}$$

As in step (6.15) the $\mathrm{d}q$ integral is

$$\int \frac{\mathrm{e}^{-q^\beta/\varepsilon_2}(1-\mathrm{e}^{-q^\beta/\varepsilon_3})}{q^\beta}\mathrm{e}^{-(q-p)^\beta}(q_1-p_1)\mathrm{d}q, \tag{6.32}$$

where $\varepsilon_3=\frac{\varepsilon_1\varepsilon_2}{\varepsilon_2-\varepsilon_1}>0$. This is in turn bounded by

$$\int \mathrm{e}^{-q^\beta/\varepsilon'}\frac{1-\mathrm{e}^{-q^\beta/\varepsilon_3}}{q^\beta}\mathrm{e}^{-(q-p)^\beta}(q_1-p_1)\mathrm{d}q. \tag{6.33}$$

We may now follow steps (6.15) through (6.27), and it is straightforward to verify in each case that the extra $\mathrm{e}^{-q^\beta/\varepsilon'}$ term allows us to replace the $O(1)$ by $o(1)$ (the o now refers to ε'). This implies that (6.31) can be made arbitrarily small by choosing ε' sufficiently small. As for the second integral

$$\int\int \frac{((1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2-(1-\mathrm{e}^{-p^\beta/\varepsilon_2})^2)}{p^{2\beta}}\mathrm{e}^{-(p^\beta+q^\beta)}p_1q_1\mathrm{d}p\mathrm{d}q, \tag{6.34}$$

we can rewrite $((1-\mathrm{e}^{-p^\beta/\varepsilon_1})^2-(1-\mathrm{e}^{-p^\beta/\varepsilon_2})^2)$ as

$$((1-\mathrm{e}^{-p^\beta/\varepsilon_1})+(1-\mathrm{e}^{-p^\beta/\varepsilon_2}))\mathrm{e}^{-p^\beta/\varepsilon_2}(1-\mathrm{e}^{-p^\beta/\varepsilon_3}), \tag{6.35}$$

and we see that we can bound (6.34) by

$$k\int \frac{\mathrm{e}^{-p^\beta/\varepsilon'}|p|\mathrm{e}^{-p^\beta}}{p^{2\beta}}\left|\int \frac{(1-\mathrm{e}^{-(p+q)^\beta/\varepsilon_2})}{(p+q)^\beta}\mathrm{e}^{-q^\beta}q_1\mathrm{d}q\right|\mathrm{d}p. \tag{6.36}$$

We have shown above that the $\mathrm{d}q$ integral is $(|p|+|p|^2)O(1)$, and that this implies that the entire integral converges. Furthermore, as $\varepsilon'\longrightarrow 0$, the dominated convergence theorem implies that the value of the integral approaches zero. Again we see that if we choose ε' sufficiently small we can make (6.34) arbitrarily small. This shows that if ε_n is a sequence converging to zero then $A(\varepsilon_n)$ converges. Thus, $\lim_{\varepsilon\longrightarrow 0}A(\varepsilon)$ exists, and we define

$$\int\int \frac{1}{p^{2\beta}}\frac{1}{(p+q)^\beta}\mathrm{e}^{-(p^\beta+q^\beta)}p_1q_1\mathrm{d}p\mathrm{d}q \tag{6.37}$$

to be this limit. This completes the calculation for components of order 2.

For a component of order $n \geq 3$ we have the following integral:

$$\int \mathrm{e}^{-\varepsilon \sum p_i^\beta} \prod_i (p_i)_1 \left(\int_{\sum c_j < T} \prod_j \mathrm{e}^{-u_j^\beta c_j} \prod_j \mathrm{d}c_j \right) \prod_i \mathrm{d}p_i. \tag{6.38}$$

We must show that this is $o(\varepsilon^{-(3n/\beta - 3n/2)})$. This would be a bit of a chore were it not that we have done almost all of the work already in the Brownian motion case. For instance, suppose we have a configuration with no isolated intervals. Then (6.38) can be bounded by (see (4.3))

$$\int \frac{\mathrm{e}^{-\varepsilon \sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1} (1 + |u_j|)^\beta} \prod \mathrm{d}p_i = \varepsilon^{-(3n/\beta - 2n - 1)} \int \frac{\mathrm{e}^{-\sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1} (\varepsilon + |u_j|)^\beta} \prod \mathrm{d}p_i. \tag{6.39}$$

We are done if we can bound this integral effectively. We know from earlier work that if β were replaced by 2 in this integral then it would be $O(\log(1/\varepsilon))^n$, which is certainly good enough. We can bound as follows using Holder's inequality:

$$\int \frac{\mathrm{e}^{-\sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1} (\varepsilon + |u_j|)^\beta} \prod \mathrm{d}p_i$$
$$\leq \left(\int \frac{\mathrm{e}^{-\sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1} (\varepsilon + |u_j|)^2} \prod \mathrm{d}p_i \right)^{\beta/2} \left(\int \mathrm{e}^{-\sum p_i^\beta} \prod |p_i| \prod \mathrm{d}p_i \right)^{(2-\beta)/2}. \tag{6.40}$$

A quick examination of the proofs of Lemmas 2, 3, 8 and 10 will show that the conclusions of these lemmas remain valid if any e^{-p^2}'s in the hypotheses are replaced by e^{-p^β}. We can conclude that (6.40) is $O(\log(1/\varepsilon))^n$, and this component is therefore sufficiently bounded. We do the same thing in the isolated interval case, with Lemma 7 replaced by Lemma 12. This completes the proof of Theorem 2.

Acknowledgement

I am deeply indebted to my advisor Jay Rosen, who suggested this problem to me, who taught me a great deal, and whose help and generosity were invaluable in completing this work.

References

[1] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.
[2] S. Edwards, The statistical mechanics of polymers with excluded volume, Proc. Phys. Sci. 85 (1965) 613–624.
[3] Rysuk Gradstein, Tables of Series, Products, and Integrals, Academic Press, New York, 1980.
[4] J.-F. Le Gall, Proprétiés d'intersection des marches aléatoires I, Comm. Math. Phys. 104 (1986) 471–507.
[5] J.-F. Le Gall, Fluctuation results for the Wiener sausage, Ann. Probab. 16 (1988) 991–1018.
[6] J. Rosen, Limit laws for the intersection local time of stable processes in $\mathbf{R}^2$, Stochastics 23 (1988) 219–240.
[7] J. Rosen, Derivatives of self-intersection local times, in: Séminaire de Probabilités XXXVIII, in: LNM, vol. 1857, Springer-Verlag, New York, 2005, pp. 171–184.

[8] S.R.S. Varadhan, Appendix to Euclidean quantum field theory by K. Symanzyk, in: R. Jost (Ed.), Local Quantum Theory, Academic Press, 1969.

[9] M. Yor, Renormalisation et Convergence en Loi pour les temps Locaux D'Intersection du Mouvement Brownien Dans $\mathbf{R}^3$, in: Sem. de Prob. XIX, in: Lecture Notes in Math., vol. 1123, Springer-Verlag, 1985.